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NAVAL RESEARCH LOGISTICS QUARTERLY

OFFICE OF NAVAL RESEARCH

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Manuscripts submitted for publication should be typewritten, double-spaced, and the author should retain a copy. Manuscripts and other items for publication should be sent to the Managing Editor at the above address. Authors will receive 50 free reprints of their published papers.

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LOGISTICS WITHOUT STORAGE

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This paper discusses ways and means for reducing storage requirements in the logistic pipeline.

The procurement, storage and distribution of supplies is such an integral part of the logistical requirements of a combat force that any idea to eliminate the storage phase is certainly an anomaly. The entire elimination of storage in the logistical pipeline of supply is neither possible nor is it safe. It is not possible because no matter how carefully plans may have been laid to achieve a smooth flow of supplies of each type, outside influences beyond control cause temporary delays to retard the flow. If this occurs, supplies are received at some points in greater quantity than they are used, so a storage requirement is created. The entire elimination of storage is not safe, for without adequate supplies in storage close to the point of use, any temporary stoppage of the supply pipeline would immediately reduce the ability of the combat force to successfully meet the enemy in battle.

After countering in the opening paragraph any suggestion that the storage phase in the logistical supply line should be eliminated, I would like for the reader to consider the other side of the coin. The excessive use of storage is obviously unwise, for the supplies in storage represent large concentrations of man-hours spent in producing these supplies, transporting them and caring for them. The supplies in storage that do not get used serve no part in the war effort. Storing up of supplies that will not be used is as much a culpable error on the part of our planners as a failure to have enough supplies. In this modern age when a nation's total effort is needed for survival, any dissipation or diversion into fruitless action borders upon the treacherous. Not only does the problem of how much storage is needed concern us, but also the problem of where the storage should be located. Ordinarily, storage is provided at points where the supplies have to be handled, such as at interchanges between means of transportation for the materials. The advantage of having the storage as near the place of use is the assurance that the temporary closing of the supply lines, or any lessening in their flow, will not be so seriously and immediately felt on the battlefield. The disadvantage of having storage points too close to the combat area is that such might be captured or destroyed by the enemy.

Another factor of the problem of storage is the snowballing of such storage operations. The need for storage facilities calls for a large part of the construction effort. The people to operate the storage facilities create added demands for support from people farther back. The more storage, the greater proportion of the nation's manpower is devoted to building facilities, carrying on supply operations and guarding such supplies with the result that fewer men are available for combat effectiveness.

In this dilemma of too little or too much, there is a course available to the planner, some things he can do which will reduce his need for storage. To the extent that he can attain the following goals, the need for storage will be reduced to the minimum essential for success of the war objective:

Self-sufficiency of the fighting units.
Standardization of supplies and equipment.
Rapid, continuous flow of supplies and equipment to match usage rates.
Effective control of inventory and scheduling.

The first two of these require but little explanation. The size of the logistic pipeline can be reduced if the combat forces are so equipped and trained as to be less dependent upon outside sources of supply. The complexity of modern weapons which has been rapidly increasing over the past decade has made the combat forces more dependent upon their rear support echelons. Some reversal of this trend may be hoped for with the greater accuracy of new control mechanisms, the greatly increased firepower of the fighting units and the potential destructiveness of the smaller sizes of atomic weapons. When one bomb or projectile has the destructive power of several shiploads of high explosive, the dependence upon supply lines and storage depots may be virtually eliminated.

The importance of standardization of supplies and equipment is evident also in the hoped for simplification of the supply and storage problem. A Navy Supply System having more than 100,000 different kinds of items offers a challenge to people who would strive to simplify and reduce the portion of the war effort devoted to keeping the combat forces supplied.

Rather than devote much space in this article to the first two of these goals, the writer prefers to turn attention of the reader to the latter two goals listed.

There are many experienced planners in the military field who would deny any possibility of substituting a rapid steady flow of supplies for the previously tried method of transporting more than the usage rate to build up supply stocks at storage points along the supply lines. The closer we can approach to the goal of rapid and continuous flows, the less will be the storage requirement. Our goal probably should be to move as far in that direction as we can without jeopardy to the combat effectiveness. It is too much to expect that a rapid continuous flow will always meet the requirements or would always be possible to achieve. We might look at the major difficulties that stand in the way of that goal. Perhaps most important is the fact that combat forces do not have steady and uniform usage rates. The requirements change from day to day and from place to place. The usage rates by a combat unit for any particular type of item for one day, might be the basis for filling the pipeline. The following day or many days later, whenever the unit receives the supplies, the quantity required is greatly changed and there is either too much or too little of that item on hand where needed. The requirements of similar units in different locations perhaps only a short distance apart or similar units assigned different missions are so different that any attempt to plan a usage rate for one based upon the usage of another unit would surely lead to serious errors.

Much as we might wish we had simpler conditions on which to plan, we do have tools we can use to meet that challenge. By maintaining effective stock control records, by use of most efficient scheduling procedures and by a speed-up in the transportation and materials - handling of supplies, we can now more closely approach the goal of "logistics without storage" than we could at any time in the recent past history of our nation.

Our problem has been likened to that of a large mail order company as Sears Roebuck, except that the number of items in the Navy Supply System is much greater. By shortening the

time taken between the point of manufacture and the point of use, we accomplish what the business man terms an increase in the turnover rate. An increase in the turnover rate means increased profit to the businessman. In a similar way, this shortening of the time means a more effective distribution system for the Navy. By speeding up the flow of finished goods, we reduce the amount in the pipeline at any one time. We decrease the labor required to handle it, decrease the storage requirements and make the distribution system more responsive to changes in the demands and usage rates of the combat forces.

The Supply - Demand Control Points set up within the Navy Supply System can control the rate of flow from the manufacturer into the System. Data on usage rates of each combat unit can be kept up-to-date by punched cards in the Electronic Accounting Machine Installations in the various S-D CP's. No longer do we need large stocking and distribution points as surge tanks in the pipeline. By keeping the manufacturer informed of current changes in requirements and the use of local storage in the vicinity of the manufacturer's plant, the means are available to keep the pipeline filled to meet changing needs. Local civilian labor and local transportation facilities are then fully utilized instead of concentrating the work of handling and shipping into large supply depots.

It is not necessary for the Navy to deal with sub-contractors of the manufacturers. Such problems of material supplies and sub-contracting should remain the concern of the manufacturer. The thousands of prime contractors with the Navy can be responsible for obtaining their materials and component assemblies.

The use of air transport can speed up the flow of goods in the pipeline. This is done not only by increased transportation speeds, but primarily by carrying supplies more directly and eliminating time spent in handling such supplies at points transferring from land to water transportation or vice versa. Many Aerial Shipping Points scattered throughout the manufacturing areas of the United States should be used; e.g., Westover AFB, Mass., NAS, Willow Grove, Penna., Brookley AFB, Alabama, and other airports at Pittsburgh, Detroit, Chicago, St. Louis and other places. Supplies could be called in to such Aerial Ports directly from the manufacturer to meet the usage rate. The shortage of air transportation facilities will always make such transport available only for high priority items. In any future wars during the next several decades, it is expected that the bulk of the supplies and equipment will have to be carried by surface transportation. High priority items can be flown from manufacturer's plant directly to overseas area in one or two days' time.

The availability of such rapid transport makes it possible to greatly reduce the quantity of many stocks in overseas depots. Spares for insurance can be reduced if transportable by air. Stock levels of all classes of supply can be held to a much lower minimum if additional supplies can be obtained quickly on call from the United States. Instead of having about 30 to 40 days of supply stored in the combat area, another 60 days of supply used to fill the transportation pipeline, and more storage at the tidewater ports, the use of air transport from the Aerial Ports in the United States to the place of use can eliminate most of the storage needs.

The large bulk classes of supplies as food and petroleum products as well as the major units of equipment and components of construction materials would still be expected to flow by surface transportation. However these supplies can usually be handled in bulk at a small cost in man-hours handling effort per ton. Whatever can be done to expedite the material handling at points of loading or unloading will aid greatly in providing the necessary logistic support without the need for abnormally large stocks of materials and their accompanying waste of manpower. The handling of petroleum products by bulk methods, particularly by pipeline,

provides the rapid movement with least use of manpower. Similarly, the mechanical devices for rapidly loading and unloading ships, trains and trucks offer possibilities for greatly speeding up the pipeline and reducing the supply stocks needed for safety levels.

The use of punched card systems for inventory control combined with rapid communications systems between the Supply - Demand Control Points and the points of usage or manufacture provides us with the tools needed for effective control over the scheduling and flow of materials.

The increase in the speed of the flow of materials from manufacturer to place of use can be obtained by greater availability of air transport for all but most common and bulky items. This will provide us with the means to have the rapid continuous flow of supplies essential to support the combat forces with the need for larger stocks of storage reduced to a minimum. We have the tools now to go a long ways toward the goal of "Logistics Without Storage."

* * *

NOTE ON THE FORMULATION OF THE THEORY OF LOGISTICS[1]

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As a preliminary step to the erection of a "theory" of logistics, the author proposes definitions of some important concepts and identifies several types of logistical situations and problems.

The lack of a rigorous and generally accepted theory of logistics at the present time is a serious impediment to making actual military logistical operations as effective as desired, although immensely complicated situations have been met, sometimes with surprisingly minor noticeable deficiencies. But, in the absence of the aid theory would render, it cannot be determined whether the solutions were really good or wasteful, and where the optimum might have been. Clearly, the future theory will have to rest on a firm empirical basis which will be won by a careful scrutiny of typical logistical operations and their descriptions in modern terms and concepts. Furthermore, it will be necessary to make as many comparative studies as possible so that the existence of perhaps several distinct types of logistical situations and problems may be discovered. Also the comparative studies — which are always called for as a quasi-experimental device when there is as yet no firm basis for a theory — must be related to the similar fields of economics and organization.

Logistical operations are primarily carried on within centrally directed social organizations and therefore fall under that part of the theory of organization that is supposed to cover that field. This is certainly true for military logistics which are, for the present, our main concern. There are also operations that carry outside the organization; e.g., when, instead of only distributing available materials and services, the organization has to go outside and, for example, must buy commodities that, after possible processing and packaging, are to be shipped in proper quantities to particular consuming units at specified dates. We may neglect those instances for the present. The logistical operations not only occur within an organization but usually require a special suborganization within the former. It will be formally identified as a special service if the logistic operations are sufficiently voluminous and complicated. Whether such a suborganization actually exists is not material for this discussion of the few basic principles.

In the following I shall try to give a brief description of what appears to me a possible formulation of the logistical situation and problem. This is not, by a long shot, the perfected kind of formulation which is the necessary first step towards a theory, but, more modestly, merely an outline which might at least lead towards a consistent language and henceforth to more adequate empirical descriptions in terms of this language. These, taken together, must precede any attempt towards a real theory.

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[1] This paper is a revised and somewhat extended version of a paper contributed to the Logistics Conference held at Santa Monica, California, in 1950. It was distributed under the above title as RAND Memorandum RM-614, dated 28 May 1951.

A logistic operation consists in the supply of definite quantities of physical means and services for activities that according to their missions consume these means and services in order that the activities be maintained at particular present or expected future rates. The supplies come from a source and must be moved, i.e., transformed in space and time, by means of transportation to the activity. A source may itself be viewed as an activity by another source that is placed higher up in the line and so on until the final source is reached. A given activity may be connected simultaneously with several sources and any one source will, as a rule, have to supply several different activities. The flow of supplies is therefore always directed towards the activity, never away from it, except for disposal which may then be viewed as a separate activity. The activity can make demands, however, and makes these effective by being recognized in the general command setup and also with regard to the rate at which it is allowed to function. A flow towards the activity can also be originated by authorities other than the activity; these are then higher operating authorities not in the chain: source - transportation - activity, but comprising it.

The logistical problem is to identify in detail the expected requirements of the activity, if the latter are described in a form that does not already contain the complete specification and enumeration of the needs. After the identification has taken place, the quantities, corresponding to the rates of the various component phases of the activity, have to be computed in such a manner as to make the entire supply plan an "optimum" (e.g., minimizing time, or costs, or tonnage used, etc., or combinations of these), under the constraint that the desired rates should be met — within stated time limits — with some specified probability. The computation will show whether the activity can actually operate, i.e., whether its operations are feasible, at the desired rates. If the operation is found feasible, the question of the optimum will arise.

Supplies of goods can be stored at various points between the source and the activity, including the latter. Services cannot be stored, except in the form of personnel which, while held idle, may be used for other purposes. Goods and services have a place and time value; i.e., the same good usually has a different value at one spot than at another, the difference being due to the difficulty and trouble of transporting it elsewhere and to the time that the transportation consumes. These value notions are familiar in economics. In the military setups it will be necessary to consider also strategic value. In economic relations place and time values are fairly stable (as a rule oranges are more desirable in the consumers' market than in the hands of the grower). But quick and frequent changes of strategic plans and situations may interfere profoundly with any established pattern of place and time values.

The values, in terms of which maximization of results or minimization of costs is to be accomplished, lead to the establishment of priorities for actions and sequences of actions (operations) within the logistical organization. (In business this is accomplished by profit calculation which is inapplicable to the military setup.) Since priorities can be set and sometimes have to be set simultaneously at different levels of the hierarchy, contradictions and inconsistency can easily develop. At present, there exists no theory of priorities, especially for the case when these have to be combined with monetary values possessed by items in the system to which no priorities are assigned.

Demands made by an activity shall be distinguished as falling into (a) "vanishing supplies," — goods which once delivered and out of the competence of the source (or of any intermediary storage point) actually disappear at the activity and do not involve the source any further; and (b) "draining supplies," — goods which, when having gotten into the possession of the activity, consume perceptibly of the activity's stock of goods and services and will, therefore, require

further supplies to be shipped to the activity. They constitute a current or future drain on the source in addition to the direct, initial shipment. Depending on circumstances, the drain will be directly on some source or first on the activity and then be indirectly transmitted back to the source.

Clearly, all services (i.e., men) belong in the second category only. Other examples are machines requiring fuel, oil, spare parts, mechanics, etc. Examples for (a) are building equipment, food, ammunition, clothing, etc. Examples for (b) are guns (which require ammunition), men (who, having to be fed, are themselves of type (b)), etc. In composite shipments care should be taken to distinguish between these two categories if one wishes to calculate the true and full impact of such transactions upon the activity as well as upon the shipping center or source. In these estimates further allowance has to be made for the different assumptions of a steady state versus changes due to war activity, the possibility that additional local supplies may be available (transforming (b) into (a)), i.e., "living off the land," etc.

Demands made by activities can in part be based on usage data which are derived from past rates of consumption, as in business. They are, however, less stable, especially where combat is involved. When new weapons are introduced there are no usage data at all so that such logistic problems as, for example, how much of different types of ammunition to store on a combat ship cannot be solved in a conventional way. The answer has to be obtained from strategical considerations that lead far beyond the narrower logistical domain.

Another profoundly important aspect of logistical operations is (1) the substitutability of some and (2) the complementarity of other goods and services. (It appears to me that these are currently too much neglected in logistics planning.) In the first case some goods can take the place of others. If the activity demands one kind of food, another might do as well; different kinds of fuel are sometimes substitutable for each other, and so on. Therefore, the source may still have a certain amount of freedom in satisfying a particular order. On the other hand, there are specific goods where no replacement is possible. Among these are specific shells, spare parts for engines, etc., etc. An order of this kind either can be filled or not, limiting the freedom of the source to possible changes in quantity. In the second case, the demand for one good is tied to the availability (somewhere along the line and at particular times) of one or more other goods. To send only guns but no ammunition (when none is at the activity) is meaningless and wasteful. Hence, it is often enough to specify one good and imply that the complementary goods will be viewed almost as component parts. The complementarity case is akin to, but not identical with, the above-mentioned of the "draining supplies." "Draining supplies" are necessarily complementary to others (namely, to those which they require over time in order to fulfill their function), but the converse is not always true, i.e., a demand for a complementary good does not necessarily involve a "draining supply." When complementarity exists, an order from the activity may restrict the freedom of the source even beyond the order itself. For example, orders may be staggered in time, but the order for good A clearly implies a simultaneous or later order for B, C, ... Of course, an entire group of complementary goods, making up a whole, may be substitutable for another, but within each the complementarity relation may hold. E.g., a stated quantity of one type of gun and ammunition may be substituted for some other quantity of another type of gun with its different ammunition.

It would be desirable that typical substitutability and complementarity relationships be worked out in advance and catalogued in far greater detail than appears now to be the case. This would increase the accuracy of foreseeing needs and the freedom for planning on the part of the source. To some extent this has been achieved by the Navy in designing hypothetical

packages, such as entire bases, etc. (Acorn, for example). But in those schemes there may be too much rigidity because substitutability within the schemes probably is not sufficiently considered.

In view of the several millions of items with which the military have to deal, it is both difficult and important to secure advance information about these properties of orders. Because of the distance of the sources from the activity it will usually be impossible to work out substitutions on an intuitive basis; therefore, special devices will have to be designed that can help to yield the desired information. This work would largely fall under the cognizance of the Munitions Board Cataloging Agency.

The smaller the activity is and the shorter the distance in terms of communications to the nearest source, the easier will it be to discover such relationships, but only at the level of local information. Wider substitutions may be possible that are entirely unknown at the lower levels but should be known to the ultimate source. Here is a field that ought to be thoroughly explored because a much greater freedom of operations may exist than is now deemed possible. These considerations show that there is a need to arrive at a comprehensive system of logistical operations in which smaller operations are embedded as special units.

Next, I mention the "compressibility" of requirements. This is not strictly part of a logistical operation, if the latter is viewed as passive in the sense that it has to execute the demands made by the activities or to compute the requirements of a strategic plan according to rigorously specified standards. Now these standards are in fact not inviolate. They are set for a level of comfort of personnel, tactical maneuverability, state of repair of engines and machines, etc., which are defined on the basis of past experience or expected needs in the case of new situations. In those estimates a minimum is seldom chosen but rather lavish provisions are often made, depending on the economic conditions of the military establishment and of the country, as well as on the fortunes of the combat operations. The knowledge of these minima would be of greatest importance if properly distributed throughout the logistical system. The source should always know the minima of the dependent activities. The minima need not be fixed, rock-bottom values to be determined once and for all; they are more likely relative minima. They can be approached by substitutions; e.g., troops may get along on fewer calories, or get them from starches instead of from the demanded proteins. The state of engine repair can be lowered, although only temporarily, so that the minimum is attainable merely for a restricted period of time. All these phenomena appear in economic life and have to some extent been described in various forms by economic theory. However, the difficulties these descriptions encounter are considerable so that much basic work needs to be done here.

It is also known that demands made by activities are frequently overstated in the hope that at least part of the demand will be fulfilled. Such experiences are encountered within any centrally controlled organization. This is additional reason why the "true" demand is always imperfectly known. The entire logistical setup is also on these grounds of a probabilistic nature which differs from one type of activity to another and shows significant variations among the various military services. Any treatment of logistics, whether purely descriptive or aiming to exact models, must take this role of probability into account.

The question of compressibility of logistic demands is of great practical significance and is closely related to the other previously mentioned characteristics of logistical operations. The determination of optima obviously depends upon whether the set standards have to be accepted. Since compressibility does not operate proportionately across the board, the interdependency between goods due to their complementarity, to "draining," etc., must be observed,

thus introducing considerable complications. If they can be resolved, it might lead to substantial economics in logistical operations but throw a heavier burden on the computational side. But this would be a desirable development.

Decisions concerning whether demands originating from an activity are true demands, whether they are compressible and ought to be compressed, what substitutions to perform, etc., make the source more than a passive organ in the whole logistic setup. It is not merely a warehouse under the control of the demands made by the activity, although such warehouses also exist. The source has a certain power of decision subject to controls exercised by higher authorities. It may therefore be viewed as a combination production and storage facility and decision point. It must make these decisions on the basis of information about its own state (including its own state of information!), the rate of the activity and its supply, including the goods in transit and about the expected future states of all these elements. Such information can only be had if there exists an information system properly embedded in the chain from source to activity. It is very likely that the greatest improvements in current logistical operations would be derived from progress in information techniques. In designing such techniques special attention will have to be paid to continuity of information patterns and storage of information in the experience of those responsible for logistical operations. The frequent changes especially of military personnel and the inadequate, and, indeed, often haphazard, record-keeping of vital components works against these requirements. Furthermore, the information system should be investigated regarding vulnerability from enemy action. Some technological perfections of information may, for example, be the exact opposite of protection of the logistical system as a whole.

There is an immediate similarity between military logistics and the logistical problems that have to be solved daily in business, notably in precise form in line production. There are also differences, the most significant of which is the size of the military problems which far exceeds anything encountered in business. Nevertheless, one may learn from a comparison, because in both fields one lacks a theory and frequently even a systematic approach. Economics has apparently neglected the study of these business phenomena altogether, but industrial engineers have made interesting contributions. In such early stages any comparative, morphological description can be of utmost importance. There is not room here to carry this out; therefore, only the following observations are made.

The simplicity of business logistics derives not only from the much smaller numbers of items even for the largest firms compared with, say, those encountered at a single Naval supply depot, but also from the fact that there is, as a rule, an instantly accessible source available in the market. Instead of having to provide a vast closed logistical system, it is possible to restrict oneself to a much smaller individual one and to supplement it as needed from the market. The overall inclusive individual military plan is replaced by a composite organization of many firms communicating with each other rapidly and accurately through the open market and the price system. Mistakes made in supplying a producing unit can be easily corrected, so that the main problem for the single firm is that of inventory control and the positioning of men and material along a production line in quantities corresponding to the controlled flow of production. Errors are, as a rule, of minor significance and seldom of the gravity of, say, supplying a battleship with wrong caliber ammunition. Also the errors made by one firm are of only minor consequence for the economy as a whole. It is doubtful that the effective form of market organization — which means that the burden for the individual firm is minimized by shifting it to the composite organization — can be used for solving military problems. Also it

should be noted that if the military setup gets out of order the disturbances are very widespread, often involving the remotest branches of the organization. Furthermore, economies with free markets have a great power of recovery from shock, an advantage that is apparently denied the current logistical setups of the military. It may be possible to design others where recovery is greater; here is an important field for investigation.

One of the chief advantages in solving problems of business logistics lies in the high degree of general information prevailing on open markets. Under most conditions a firm can easily determine to what changes its supply situation can be subjected. A price change is a signal for a firm to adjust input, output, and/or its own prices. All this information is practically offered free by the market. A military activity, on the other hand, making its requirements known to a source is generally not in a good position to know well in advance how quickly and to what extent its demands will be met. The firm operates only under the constraints of price and its own funds. I believe that deeper analyses of the problems of military logistics will show that the most difficult and most important aspects lie in the field of information and in the flow of messages and papers. It would require a separate treatment to enter upon this matter. At any rate, for the economy as a whole there exists a wide, universal information flow similar to that provided by the blood stream which carries general information to all parts of the human body, while the closed military system essentially provides separate channels and signals, akin to the nervous system. The latter, if it works well, is highly efficient, but complicated and vulnerable. It is probable that military systems in general, and military logistics systems in particular, do not provide for themselves enough general information for efficient operation. There is, in any large organization, a great need for a flow of information that is not specifically directed to any part of it in order to secure spontaneous adaptive response to changing conditions.

Another distinguishing feature lies in the speed with which transitions from a low state of activity to a high state occur. In business the consecutive differences in states are small or, when large, may be of a seasonal nature and are therefore well foreseen. Business production sometimes does undergo big changes in volume, but always well spread out over time compared with the sudden, almost instantaneous changes expected frequently of military logistical operations. In the latter case advance information is often totally lacking, e.g., because of the dependence on changes in strategic plans or on the fate of battles. Furthermore, the amount of change involved even within days may exceed anything known in the random changes of business production. This difference between military and business logistical situations is of major significance. It is not sufficiently appreciated by many critics of military logistical operations who, for example, do not see that a battle may transform within hours a condition of shortage of supplies into wasteful surplus or vice versa. This virtually never happens in business.

The best illustration of a fairly well investigated problem of business logistics and its empirical solution — not necessarily optimal — is offered by line production. There, principal and subsidiary conveyors consume vast quantities of materials and services which have to be held ready at precise moments of time at specified points in definite amounts. There is often a considerable variation in the final product which, of course, complicates the logistical situation. The fact that these systems exist and function well is an indication that at least empirical solutions have been found and often for significantly different conditions. It appears likely, therefore, that some theoretical penetration of these problems is possible. Such theory would offer a kernel for the theory of the wider problems of military logistics. It can only be attempted if a large amount of descriptive work has been done.

The immediate task appears to me to lie in providing adequate descriptions of typical military logistical situations and a formulation of the problems they actually contain. In this the military must help, especially officers and other operating personnel in the field itself. Any kind of functioning or malfunctioning of the various organizations must be brought out into the open. While such descriptions may exist, they are either inaccessible or are not being used in setting up models that would permit modern analysis. A study of a Naval supply depot conducted at Princeton has shown how difficult it is to discover the localization of the limitational factors, to describe the interaction of the various component parts, and to define and measure the capacities at which the Base can operate. Only after this side of the problem is straightened out can one collect the proper statistics and then formulate a model that may produce numerical results in the attempt to find an optimum solution. The difficulties in giving descriptions are formidable but they must be faced. One of the principal obstacles is that the operating personnel are seldom able (nor have the time) to do the describing of their own activities. This corresponds to the inability of an ordinary human being to say what goes on in his own body, although his experience with living is considerable. It corresponds also to the inability, or at least difficulty, of the businessman to describe and analyze his actions and to state the why and how of his decisions. Next, the military personnel in charge has its own terminology which differs from that of the economist, making complicated translations necessary in order to establish a consistent language, which is now lacking. Descriptions should be made of bases, the transport systems, the decision points for inventory control, the consumptive activities, such as air fields and their units, etc. Although much is known about all this, the knowledge is not in the form in which it is required for purposes of logistical analysis which ultimately will have to be in mathematical form.

Substantial efforts have been made to establish computational procedures for determining military requirements in complete detail; for example, when a large fixed amount of money is available that has to be translated into specific orders spread over a lengthy future period. Consider an appropriation of X billion dollars to be spent over some years on aircraft. After the strategic decision has been made how many first line planes of what kind to buy, one has to spell out the precise number of spare parts, runways, hangars, pilots, canteens, hospital-beds, storage tanks, etc., etc. All these have to be ordered and produced in the right quantities, delivered at the proper places in the correct sequence and amounts. This is a typical logistical operation. Since the same program can normally be fulfilled — if at all — in a variety of manners, there is the problem of the selection of the optimal program. Methods have been developed, especially by the Air Force Project SCOOP, that allow the advance computation of the requirements well ahead of the time of placing the orders and the date of delivery. Since the details are of a highly technical character we merely refer to the existence of these techniques. The better acquainted other services become with the achievements of the kind of methods developed by SCOOP and others, the greater will be their readiness to cooperate and to provide that full and free flow of information that is indispensable.

The preceding remarks make it clear that a theory of (military) logistics will be difficult to establish and complicated in nature. It will have to incorporate elements from other fields, such as economics, management, mathematics, etc. It must be firmly empirical; it must be operational. Even though we do not have, at present, more than some elements of this future theory, we can — and, indeed, must — formulate these and other properties it will have to possess in order to satisfy. A theory of logistics will have to tell us how to determine equilibria in logistical situations, how to make transitions to new equilibria at vastly different

levels of operations, how to compute optima in all cases, and how to design novel logistical organizations. These are rather formidable demands, but they are not unreasonable. Yet they show us how far we are still removed from the desired state of our knowledge. Clearly, it will take many good minds and much time and effort to bring us nearer to the goal.

* * *

ORGANIZED DECISION MAKING

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Some aspects of the phenomena associated with decision making by an organization are discussed. Their interplay is illustrated by a simple example.

1. INTRODUCTION

As a rule, an organization faces various situations, and its structure must be equipped to handle all of them in some fashion. What may be considered a good structure may very well be a compromise among many structures, each individually tailored for specific situations, whose composition depends on the frequency and importance of the individual situations faced.

Thus, in general, an analysis of an organizational structure might take the form of a series of analyses of its characteristics in specific situations, and a subsequent evaluation of its effectiveness over an expected distribution of these situations.

This paper is to develop a representation of some of the phenomena associated with decision making by an organization. The representation includes the allocation of responsibility, and reservation of authority for an instance of organized decision, and the processing and transmission of the information and policy required for an optimal decision by the organization.

On the basis of this representation, the effort required of the organization in processing and transmitting sufficient information and policy for an optimal organized decision can be expressed as a function of the allocation of responsibility and reservation of authority for this instance of decision. Relative to this particular instance and a valuation of organizational effort, this analysis permits the evaluation of an organizational structure as specified by its allocation and reservation characteristics.

The ideas of the representation and its use are quite simple, and they might be most easily introduced by the following entirely unbelievable example.

2. EXAMPLE

Imagine, if possible, a slaughterhouse with a daily capacity of one cow and one pig. This establishment processes (either or both) a live cow or live pig into a dressed carcass. The slaughterhouse has no stockyard or refrigeration facilities, and cannot hold an inventory of live animals or carcasses over.

Daily, the prices of cows and pigs, both live and as dressed carcasses, undergo various fluctuations. The owner of the establishment learns the day's prices each morning. The market is such that cows or pigs, dressed or live, may be purchased or sold at will.

On any given day, let:

p_c be the price of a dressed cow

p_p be the price of a dressed pig

q_c be the price of a live cow

q_p be the price of a live pig

At the same time let the cost characteristics of the operation be:

c_f The costs independent of the operation of the slaughterhouse (say, rent, etc.)

c_s The costs, over and above c_f, of setting up the slaughterhouse for the processing of either or both a cow or pig (say, labor, etc.)

c_c The costs, over and above c_f and c_s, incurred in processing a cow, whether or not a pig is processed (say, labor, etc.)

c_p The costs, over and above c_f and c_s, incurred in processing a pig, whether or not a cow is processed (say, labor, etc.)

The problem facing the slaughterhouse is whether or not to process either a cow or a pig, or both. We take maximum profit as the motive of operation. (For this discussion, suppose the question of retaining the slaughterhouse is settled — some sort of long-term consideration based on expected future market prices would seem to be required for this.)

The owner is faced with finding the maximum profit attainable under the day's prices — that is, he considers the possibilities, with accompanying outcomes of:

Processing nothing: $-c_f$

Processing a cow only: $p_c - q_c - c_c - c_s - c_f$

Processing a pig only: $p_p - q_p - c_p - c_s - c_f$

Processing cow and pig: $p_c + p_p - q_c - q_p - c_c - c_p - c_s - c_f$

and selects a possibility with the maximum outcome.

We note, for later comparison, that this calculation consists of forming four expressions out of essentially seven pieces of information ($-c_f$ occurs in each expression). Here, the owner is the entire "organization" concerned in the daily decision.

Now, consider a new organization defined by the following specifications:

The members are the owner, a cow foreman, and a pig foreman, where

(a) The owner is responsible for setting up slaughterhouse for processing,

(b) The cow foreman is responsible for decision to process a cow, if facilities are set up,

(c) The pig foreman is responsible for decision to process a pig, if facilities are set up.

We can express the outcome of an organized decision arithmetically by adopting the convention of assigning decision variables to the decision makers as follows:

$d_o = 0$ means owner does not set up slaughterhouse,

$d_o = 1$ means owner does set up slaughterhouse,

$d_c = 0$ means cow foreman decides not to process cow,

$d_c = 1$ means cow foreman decides to process cow,

$d_p = 0$ means pig foreman decides not to process pig,

$d_p = 1$ means pig foreman decides to process pig.

(If d_o is zero, then d_c and d_p must be made zero if not zero already.)

The day's profit is determined by the choices made by the three members. Let it be

$$P(d_o, d_c, d_p) .$$

In order to process a cow or a pig, both d_o and d_c or d_p must be 1. Thus, we can write

$$P(d_o, d_c, d_p) = d_o d_c (p_c - q_c - c_c) + d_o d_p (p_p - q_p - c_p) - d_o c_s - c_f$$

which may be rearranged into

$$P(d_o, d_c, d_p) = d_o [d_c (p_c - q_c - c_c) + d_p (p_p - q_p - c_p) - c_s] - c_f .$$

Now, consider how the members can behave so as to guarantee the maximum profit attainable. One way, which is easily verified, is the rule

(a) The cow foreman examines $p_c - q_c - c_c$. If this expression is positive, set $d_c = 1$; if negative, set $d_c = 0$; if zero, d_c arbitrary. Relay the number $r_c = \max (0, p_c - q_c - c_c)$ to owner.

(b) The pig foreman examines $p_p - q_p - c_p$. If this expression is positive, set $d_p = 1$; if negative, set $d_p = 0$; if zero, d_p arbitrary. Relay this number $r_p = \max (0, p_p - q_p - c_p)$ to owner.

(c) Owner examines $(r_c + r_p - c_s)$. If this expression is positive, set $d_o = 1$; if negative, set $d_o = 0$; if zero, d_o arbitrary. (The day's profit will then be $\{\max (0, r_c + r_p - c_s) - c_f\}$.)

A feature of interest in this particular formulation of a decision rule is that, whereas a single owner was required to evaluate four expressions of seven pieces of information, the organization has the property that each individual is required to only evaluate one expression of three pieces of information. In this sense, the total problem has been broken up into an integrated set of three simple problems by means of "organizing" in this particular way, and specifying the accompanying decision criteria.

3. A FINITE REPRESENTATION OF ORGANIZED DECISION MAKING

In the example, the critical features of the three man organization representation were:

(a) A finite set of members was chosen.

(b) Each member was given a finite set of alternatives from which a choice was to be made — these sets of alternatives, as a whole, represented the entire freedom of choice available to the organization.

(c) The outcome, as a real number, was written as a function of the sets of choices (one by each member) possible. The function happened to be of arithmetic character — all that is needed, however, is a correspondence between each set of choices and an outcome.

(d) An order in which a set of organizational operations by the members is to be made was chosen.

(e) A decision rule was formulated which guaranteed the largest possible outcome by means of successive operations by the members in the order chosen. In the instance of operation by a member in a decision rule, two distinct entities occur. Given the set of previous operations, for each operation to follow there occurs a hypothetical set of choices (one from each following member) and the operation defines

(1) A contingent outcome — the largest possible outcome attainable.

(2) A contingent set — the set of choices by the member which attains this largest possible outcome.

Both of these entities depend on the set of choices yet to be selected, and we can think of them as functions of these choices.

From these features, we make the interpretations:

Allocation of Responsibility - The set of choices allocated to each member and the function relating sets of choices to the outcome are interpreted as the allocation of responsibility.

Reservation of Authority - The order in which operations are to be made is interpreted as the reservation of authority.

Processing of Information - In each operation, the construction of the contingent outcome, as a function of the remaining choices is taken to be the processing of information.

Transmission of Information - The relay of the function representing the contingent outcome from each member to the member responsible for the next operation is taken to be the transmission of information.

The last member to perform an operation constructs a contingent outcome and contingent set which depend on no other choices — i.e., they are constants. This last contingent outcome is to be the actual outcome of the situation. A choice from the contingent set is taken to be the beginning of policy formation for the situation. Since the next to last contingent set depends only on the last choice, it can be evaluated when the last choice is determined. Similarly, when the last two decisions are determined, the second from the last contingent set can be evaluated, and so on.

Transmission of Policy - The relay of successively determined choices (in the opposite direction of information) is interpreted as the transmission of policy.

Processing of Policy - The evaluation of the contingent set on receipt of the necessary set of determined choices and the selection of an arbitrary choice in it is taken to be the processing of policy.

Over-all, the interpreted process is as follows: In the order of operations selected, each member constructs a contingent outcome and a contingent set, both as functions of choices yet to be made. The contingent outcome is relayed to the member responsible for the next operation, and the contingent set is retained for later use. When the sequence of contingent outcomes has been constructed successively and relayed to the last member, he selects a choice from his contingent set (which depends on no other choices). Now, successively (in the opposite order of choice) each member evaluates his retained contingent set by means of choices relayed to him, and in turn, relays a choice from his contingent set, along with the rest, to the next member. When this process is completed, each member uses his choice so selected. The outcome of the situation is the contingent outcome of the last member.

The question may arise as to whether this final outcome is the maximum attainable, and whether the order of choice affects this outcome. The answers are yes and no, respectively. That is, the process described will provide the maximum attainable outcome, no matter what order of choice is chosen.

However, the order of choice does affect the contingent outcomes and contingent sets derived in the process, and in the interpretation the entire effort of the organization in arriving at the organized decision is expended in the construction and transmission of these entities. The sequence of contingent outcomes and contingent sets, thus, on the one hand, are determined by the nature of the functional outcome (allocation of responsibility) and the order of choice (reservation of authority), while on the other, determine the actual work content of the organization in making an optimal decision.

4. USES OF THE REPRESENTATION

In some relatively simple situations of decisions in a typical hierarchial organization, where each person, save the highest executive, reports to exactly one other person, the idea of the example can be directly extended by:

(a) allocating costs to the lowest level positions which control them,

(b) allocating income to the lowest level position which participates in the income producing action,

if, indeed, such allocations can be made meaningfully in an unambiguous way. Having done this, the representation displays clearly:

(a) A location for each type of individual decision involved.

(b) The exact information requirements for each individual decision.

(c) The communication (in information and policy transmission) required for the organization decision.

The information specification suggests a cost accounting convention, as part of the total information structure of the organization, in the form of costs allocated to decision points rather than to products, etc. It is quite easy to see that no other type of information specification can guarantee optimal individual decisions at as low a level throughout their locations in this particular representation. That is, the decisions are as low as possible in the hierarchy and require exactly the communication content and structure specified.

As a simple example of the extension, we might consider our slaughterhouse "merged" into a parent organization. Then the "owner" of the slaughterhouse reports upwardly in the same fashion as the foremen report to him.

To give a further idea of the differences of accounting conventions, we might consider the slaughterhouse profit as a function of "allocated product costs" — say the indirect costs are allocated to the cow and pig operations in the ratio a_c and a_p, where $a_c + a_p = 1$. Then it can be verified that $P(d_o, d_c, d_p)$ takes a "form" such as:

$$P(d_o, d_c, d_p) = d_o d_c \{p_c - q_c - [a_c + (1 - d_p)a_p](c_s + c_f)\} + d_o d_p \{p_p - q_p - [a_p + (1 - d_c)a_c](c_s + c_f)\} - (1 - d_c)(1 - d_p)(d_o c_s + c_f)$$

where the allocated product costs are

$$[a_c + (1 - d_p)a_p](c_s + c_f) \text{ and } [a_p + (1 - d_c)a_c](c_s + c_f)$$

for the cow, pig operations respectively.

With this information a specified requirement, the decisions d_o, d_c, d_p cannot be separated in any way as before — it is not possible to push the decisions down and guarantee correct ones with this kind of information.

Of course, the extension of the example as used is quite simple, and this means it requires simple situations to be literally useful. Most situations facing organizations are not simple enough to be "factored" up as such. In particular, a failure of the extension will occur when income producing or cost producing actions depend on joint decisions at points not in a direct line of the hierarchy.

5. CONJECTURES

It is a trivial matter to set up mathematical notation for the representation. Given the allocation of responsibility in the form of a function

$$F(d_1, d_2, \ldots, d_n), \; d_i \in D_i, \; i = 1, 2, \ldots, n,$$

the reservation of authority is an ordering on the integers $(i_1, i_2, \ldots, i_n)$, and the model of operations is the mathematical operation

$$\max_{d_{i_n} \in D_{i_n}} \cdots \max_{d_{i_2} \in D_{i_2}} \max_{d_{i_1} \in D_{i_1}} F(d_1, d_2, \ldots, d_n)$$

viewed as progressing one step at a time. At each step a variable is suppressed and a new function (contingent outcome) of one less variable is formed. At the same time, this step places constraints on the variable suppressed and these define a set (contingent set) {[2], pp. 89}.

We might regard this as a convenient way of putting the preceding representation into symbolic form. However, aside from the commutivity of the max operators (inconsequence of order of choice insofar as the outcome is concerned) there seems to be little content of useful mathematical theorems available without more specialization of the function itself.

A source of additional insight in organized decisions seems likely, however, in two possible directions:

(a) In the construction of $F(d_1, d_2, \ldots, d_n)$ itself. We might regard this function as a "normalized" form of a situation. In arriving at this form, out of a real situation, choices are available in the sets, D_i, and the form of F.

This suggests an attempt to formalize an "extensive" description of a situation independently of the organization eventually to be realized so that various structures of organizations can be compared by means of the transformations into normal (organization dependent) forms and evaluated along the lines suggested above.

(b) In the normalized form to study the problem of maximizing the outcome with incomplete information between members. Recent work in statistics may be applicable, especially in factorial analysis and response surfaces.

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A FORMULA FOR DECISIONS ON RETENTION OR DISPOSAL OF EXCESS STOCK[1]

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The author proposes a formula for use in deciding whether to retain or sell to the highest bidder large stocks of items of material that appear to exceed demands in the immediate future. Estimates of expected demand, storage cost, percentage return from disposal sales, probability of obsolescence and interest charges are the basic factors employed in the formula. Given these data for an item, and that certain assumptions used in the derivation of the formula are satisfied, one may determine the optimum level of stock to retain.

1. INTRODUCTION

When a government agency or a private corporation finds that it has large stocks of various items of material excess to its expected needs in the immediate future, the problem of deciding between retention of the material or disposal to the highest bidder arises. This report presents a formula from which to determine how much stock should be retained.

The factors which should be considered in making these decisions have been brought together in the formula below. All the costs of each alternative, both real money costs and the intangible costs of having too little too late, can be represented in the variables of this formula. These variables represent our best guesses about the future need for the item, the costs of holding it, the chance that it will deteriorate, and its price and availability if needed in the future. Thus, the benefits of holding material can be compared with the costs and losses which will probably be incurred. The formula is intended to provide a clear and easy-to-use scientific procedure for reducing estimates of future probabilities to specific numbers yielding consistent decisions.

The formula represents a balance between: (a) the cost of storing the material for some years, considering the chance that it may become obsolete and worthless, and (b) the cost of purchasing it again some years hence when and if needed, if present surpluses are sold by disposal action today. Comparing these costs, we find a break-even point of time, x years in the future. Stock sufficient to meet expected needs up to that time should be retained; any excess over that amount should be sold.

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2. THE FORMULA AND ITS DERIVATION

Assumptions

For the purposes of this analysis, the following assumptions are made. These assumptions permit us to write a relatively simple formula and derive a set of typical values. The assumptions may be relaxed to deal with more complex cases later; a discussion of these cases is presented in the latter part of this report.

(a) It is assumed that annual demand in the future is known and constant.

(b) It is assumed that the most likely future price for new procurement of an item is its present unit price. In other words, out best guess of how much the item will cost x years from now is its present price.

(c) It is assumed that the general price level, the cost of storage, the interest rates, and the probability of the item becoming obsolete will also remain constant at values which can now be estimated.

(d) It is assumed that the value of material when used by the government agency or firm is its present unit price v; that its disposal value if sold as surplus may be represented as a fraction D of its unit price v; and that there is a finite probability p that its value will drop to zero in any one year because the material becomes completely obsolete or deteriorates. Only these three representative values of the material - v, D, and 0 - will be considered.

Definition of Variables

Let us define the following variables:

S = number of units of stock of an item on hand

U = expected average annual demand in future years

$x = \frac{S}{U}$ = number of years for which a stock of S will meet expected demand

v = value of material, assumed to be the present standard price per unit

D = fraction of present unit price of material which will be realized in disposal sales - e.g., if 15 cents on the dollar, D = .15

i = interest rate

p = probability that material will become obsolete and worthless in any one year - e.g., if p = .01 there is one chance in a hundred that the value of the material to the stocking activity will drop to zero during the forthcoming year

r = annual storage cost of material, expressed as a fraction of the dollar value of the material - e.g., r = .01 means that it costs 1 cent to store material worth one dollar for one year

C_R = total cost of retention of stock

C_D = total cost of disposal of stock

Economic Retention Period Formula

Equation (1) uses these cost factors to find the value of x which will equalize the costs of retaining and disposing of stock. The value of x represents the number of years for which stock should be held and is defined as the economic retention period. Since we have assumed a known and constant value for U, the quantity of stock S_R which should be retained is proportional to and determined uniquely by the value of x.

$$(1) \quad D(1+i)^x - (1-p)^x + [r(1-p)(1+i)^x + r(1-p)^2(1+i)^{x-1} + \ldots + r(1-p)^x(1+i)] = 0$$

The value of x defines the break-even point in time. Stock sufficient to meet expected demands until that time should be retained; any additional stock should be disposed of.

Derivation of the Formula

The derivation of this formula may be explained in terms of the several cost factors involved. This section may be omitted by any reader not interested in the economic justification of the formula.

Let us look at the costs which will be incurred under the two alternatives of retaining excess stock or disposing of it.

First we shall look at the costs of retention of stock. We shall define C_R as the costs which will be incurred, from the present time (t_o) to the time x years hence (t_x), if a given quantity of stock to meet demands beyond t_x is retained in storage. These costs C_R are of two kinds: the storage costs and the loss in value if the material should become worthless.

The storage costs we shall define as r dollars per year. But there is a small probability p that the material will become worthless by deterioration or obsolescence and have to be thrown away. If this should occur, storage costs will no longer be incurred; therefore, the storage costs from t_o to t_x should be reduced each year. This probability may be expressed as $(1-p)^t$. Now we must make one more adjustment to arrive at the storage cost portion of C_R. If the material had not been stored, the money used for storage costs could have been increasing in value at compound interest rates; therefore, the costs of storage must include the interest otherwise earned, as represented by $(1+i)^t$ for each year. The storage costs are therefore $r\,(1-p)\,(1+i)^x + r\,(1-p)^2\,(1+i)^{x-1} + \ldots + r\,(1-p)^x\,(1+i)$ for the period of years from t_o to t_x.

The second kind of cost which may be incurred if the material is retained is the loss if the item becomes obsolete or if the material deteriorates. In this case each dollar of value of the material will be lost. The probability of this occurrence we represent by p in each year so that the loss is $1 - (1-p)^x$. Therefore,

$$C_R = 1 - (1-p)^x + r\,(1-p)\,(1+i)^x + r\,(1-p)^2\,(1+i)^{x-1} + \ldots + r\,(1-p)^x\,(1+i) \tag{2}$$

It may be seen that these costs C_R will increase as x increases since p, i, and r are all less than 1.

Next we shall look at the costs of disposal of stock. Here the alternative is to dispose of the given quantity of material now and procure it again at time t_x to meet demands then. We shall call this alternative C_D, the cost of furnishing a given quantity of an item at time t_x. The major cost is the repurchase of the material at time t_x, at an assumed unit price of v, or \$1 per dollar value of material. But this cost must be reduced by the return from the disposal sale, D, and this amount will have increased in value at compound interest rates until t_x. Therefore,

$$C_D = 1 - D\,(1+i)^x \tag{3}$$

The value of this expression decreases as x becomes larger; it moves in the opposite direction from C_R, and we may therefore expect the values of C_R and C_D to become equal at some value of x.

We have now defined the functions of C_R and C_D so that we can tell the total costs of either retaining or disposing of a quantity of stock S. We wish to find a value of x which will

equalize these costs, i.e., a point of time t_x at which we are indifferent between retaining stock and disposing of it. At values less than this, the cost of disposing of stock will be higher; at values greater than this, the cost of retaining stock will be higher. We wish, therefore, to find the value of x at which

$$C_R - C_D = 0 \tag{4}$$

Substituting the functions above, we get:

$$1 - (1-p)^x + r\,(1-p)\,(1+i)^x + r\,(1-p)^2\,(1+i)^{x-1} + \ldots + r\,(1-p)^x\,(1+i) - 1 + D\,(1+i)^x = 0 \tag{5}$$

Rearranging, this becomes equation (1):

$$D\,(1+i)^x - (1-p)^x + r\,(1-p)\,(1+i)^x + r\,(1-p)^2\,(1+i)^{x-1} + \ldots + r\,(1-p)^x\,(1+i) = 0 \tag{1}$$

In this section we have written mathematical expressions enabling us to determine the cost of retaining excess stock and the cost of disposing of excess stock, where excess is defined as stock excess to our expected demand for x years. By equating these two cost functions, we may find the value of x at which the costs of retention and disposal are equal. Let us call this particular value $\bar{x}$.

Then, since $x = \frac{S}{U}$ and U is given in a particular case, we may find the particular value $\bar{S}$ for this value $\bar{x}$. The value of $\bar{S}$ is the number of units of excess stock which should be retained. Any stock on hand above that quantity should be disposed of immediately.

Solution of the Formula

Equation (1) may be solved for x as follows:

$$D\,(1+i)^x - (1-p)^x + r\,(1-p)\,[(1+i)^x + (1-p)\,(1+i)^{x-1} + \ldots + (1-p)^{x-1}\,(1+i)] = 0 \tag{6}$$

$$D - \frac{(1-p)^x}{(1+i)^x} + r\,(1-p)\left[1 + \frac{1-p}{1+i} + \frac{(1-p)^2}{(1+i)^2} + \ldots + \frac{(1-p)^{x-1}}{(1+i)^{x-1}}\right] = 0 \tag{7}$$

Summing the geometric series in brackets,

$$D - \left(\frac{1-p}{1+i}\right)^x + r\,(1-p)\,\frac{1 - \left(\frac{1-p}{1+i}\right)^x}{1 - \left(\frac{1-p}{1+i}\right)} = 0 \tag{8}$$

The terms containing $\left(\frac{1-p}{1+i}\right)^x$ are then collected,

$$D - \left(\frac{1-p}{1+i}\right)^x + \frac{r\,(1-p)}{1 - \frac{1-p}{1+i}} - \left(\frac{1-p}{1+i}\right)^x \cdot \frac{r\,(1-p)}{1 - \frac{1-p}{1+i}} = 0 \tag{9}$$

$$(10) \qquad \left(\frac{1-p}{1+i}\right)^x \left[1 + \frac{r\,(1-p)}{1 - \frac{1-p}{1+i}}\right] = D + \frac{r\,(1-p)}{1 - \frac{1-p}{1-i}}$$

$$(11) \qquad \left(\frac{1-p}{1+i}\right)^x = \frac{D + \frac{r\,(1-p)}{1 - \frac{(1-p)}{(1+i)}}}{1 + \frac{r\,(1-p)}{1 - \frac{(1-p)}{(1+i)}}}$$

$$(12) \qquad \left(\frac{1-p}{1+i}\right)^x = \frac{D + \frac{r\,(1-p)}{\frac{(i+p)}{(1+i)}}}{1 + \frac{r\,(1-p)}{\frac{(i+p)}{(1+i)}}}$$

$$(13) \qquad \left(\frac{1-p}{1+i}\right)^x = \frac{D + \frac{r(1-p)\,(1+i)}{(i+p)}}{1 + \frac{r(1-p)\,(1+i)}{(i+p)}}$$

$$(14) \qquad \left(\frac{1-p}{1+i}\right)^x = \frac{D\,(i+p) + r(1-p)\,(1+i)}{i+p+r(1-p)\,(1+i)}$$

$$(15) \qquad x = \frac{\log \frac{D\,(i+p) + r(1-p)\,(1+i)}{i+p+r(1-p)\,(1+i)}}{\log \frac{1-p}{1+i}}$$

Tables for the value of x for various combinations of values of D, i, p, and r have been prepared for NSRDF by the Applied Mathematics Laboratory of the David Taylor Model Basin, and are available in the report distributed by NSRDF.

3. HOW TO USE THE FORMULA

To use this formula in retention and disposal decisions, it is necessary to estimate the values of the variables D, i, p, and r for each item or group of items. The value of x, the number of years stock is to be retained, may be obtained directly from the tables or by solution of the formula. We thus determine what part of the stock, if any, should be disposed of. Let us now look more closely at these variables, using Navy stock items as examples of use of the formula.

It must be realized immediately that exact values of these variables cannot be found; use must be made of rough estimates based on available demand and cost data. Because these

symbols look strange, the operator's reaction may be that their values cannot be even estimated and therefore the formula is useless. But retention and disposal decisions are being made, by every agency every day, on the basis of individual guesses and judgments. The decisions now being made are based implicitly upon guesses which the stock reviewer makes about how likely it is that the material will become obsolete, how much money a disposal sale might net, and how much it costs the Navy to store an item. Use of the formula will make these estimates explicit and insure that a scientific and logical calculation is made. It will insure consistency in these decisions when applied to different groups of items by different persons. It will enable the agency to collect better cost data and improve these decisions, because the kind of cost data needed is specified. It will enable an agency to pinpoint the variable which was estimated incorrectly if the decisions prove faulty. Therefore, the fact that these variables cannot be known perfectly does not decrease the merits of the formula. The substitution of a logical calculation should be a marked improvement over present ad hoc methods for retention-disposal decisions.

The Disposal Percentage

The value of D, the percentage return on disposal, will differ widely from one type of material to another. The disposal percentage must be interpreted as the net return to the Navy after deducting the costs of sale of the material and costs of moving and delivering it. Even for items of general stores material, where a wide range of commercial uses causes buyers to pay close to the Navy standard price, the costs of sale mean that the value of D can seldom go over 75%. For many items of general stores, useful mainly for the Navy and made to Navy's higher specifications, a net return in the neighborhood of 15% to 25% is more likely. For items of specialized use like electronic tubes or aviation parts, many of which have only scrap value when sold, the costs of sale often come close to equaling the disposal price and the net return may be as low as 2%, 3%, 5%, or 10%. For a good many items located at outlying bases or overseas, the cost of transportation back to markets would be too great and the value of D is zero. If the formula indicates disposal of some such material, it would, of course, be consigned to the scrap heap.

Each Navy supply demand control point should have data on disposal percengages for material under its cognizance, and procurement officers and stock reviewers can usually add their experience to the determination of proper percentages. It might be worthwhile to select three or four typical values, say 0%, 5%, 20%, and 75%, and to classify each item or group of like items for one of these values. Thus, all anchors located in the United States might be classified at 20% disposal value, and all electronic tubes at 5%. Cable located at Norfolk and Oakland might be estimated at 75% disposal value since sales there are frequent and cheap; whereas, cable located at air stations remote from buyers might be classed at only 40% disposal value. It is understood that all major Naval activities report gross and net returns from surplus sales to the Office of Naval Material regularly, and these returns may be useful in selecting proper disposal percentages.

Use of the term "disposal sale" here does not mean that an excess item held by a Navy activity need be actually sold. In many cases items found to be in excess by one government activity or agency are needed by another agency. In this case, D should be the percentage of standard price at which the item is transferred (for Naval Stock Account material, this is frequently 50%), less the costs of transfer. These costs of transfer include the paperwork of preparing and circulating catalogs, the packing of material for shipment, and the transportation costs. These costs bulk large in many cases, especially where small amounts are concerned.

In many cases the problem is not one of retention or disposal of unused stocks but of used items of equipment. In these cases, the formula remains unchanged and the only adjustment is to lower the standard unit price of the excess stock from its new cost, since its value to the activity has depreciated with use. For example, ten new fork lift trucks may be worth $10,000. After eight years of use, their remaining useful life is shorter. They may then be estimated to be worth only $5,000 in future expected service. The future expected demand U for fork lift trucks we have assumed to be known. But this was demand for new fork lift trucks. If used fork lift trucks are worth only half the price of new fork lift trucks, then we may simply count two used fork lift trucks in our stock as the equivalent of one new fork lift truck. The disposal percentage D will still be the percentage of the present dollar value of $5,000 that disposal sale would realize; for many items of used equipment D will probably be fairly high, perhaps 40% - 50%. The general principle remains the same, however, and the formula is therefore applicable to all types of items in various conditions.

The importance of transportation cost from the warehouse to the buyer must not be neglected. If surplus sales are made f.o.b. the Navy warehouse, no transportation costs are incurred. But, if the material is transferred within the Navy, the expense paid out of the Navy transportation appropriation must be considered in estimating D, even though the Bureau or activity allotment is not affected.

The decisions on redistribution of material between Navy activities, when one depot has an excess and another depot requires the same item, may be analyzed by formulas similar to the retention-disposal formula above. In these cases the transportation and handling costs are often predominant and it is sometimes better to purchase new material rather than move an excess stock. A report is being prepared by NSRDF proposing a method for these decisions based upon costs of procurement, handling, storage, paperwork, and transportation.

The Interest Rate

An interest rate i of 3%, the rate paid on U. S. Government Series E bonds, would seem to be the proper rate for government agencies. But, since some Treasury paper carries rates as low as 1%, the tables have been made up to show the value of x for other values of the interest rate. Commercial firms borrowing money may pay as much as 6% interest, so this value is also shown.

A Navy agency may state that it pays no interest charges since the money to pay for the material has already been appropriated. This argument is not correct since money freed from investment in inventory may be used for other productive purposes, or used to reduce the size of the Federal debt. On the whole, a value of 3% seems most reasonable for Navy activities.

Probability of Obsolescence and Deterioration

As with the disposal percentage, this probability p will depend largely on the type of material. This probability is, in the formula, the chance that the material will become obsolete in one year. For general stores items like nuts and bolts which will be used for centuries, the probability of the material becoming obsolete is very low. But many common Navy items will probably have been replaced by better models or finer materials in 50 years; a probability of .02 per year for such items seems reasonable. For many aviation parts peculiar to a particular aircraft, the life of the plane sets an effective limit to the useful life of the part. If the service life of the plane is 10 years, a probability of at least 10% is justified. Many items are subject to deterioration, and for some items it is known that better models will be available in a few years. For both these classes of items the probability of obsolescence should be raised to 20% or even 30%.

Storage Cost

The storage cost per year, r, should include the variable costs of warehousemen and guards, and in some cases the fixed costs of buildings and land where the material is stored. It is not always correct to say that the warehouse would be there anyway and is therefore costless. If the item in question were sold as surplus, the space would be available for other material. The Navy is usually building or acquiring new storage space in some location and in time of war expansion of facilities is widespread. The important question in retention-disposal decisions is whether the activity holding the stock will make use of the space vacated. One rough way may be suggested for deciding whether to include fixed and overhead costs in the storage cost figure. Each activity makes a storage utilization report monthly showing the percentage of storage capacity that is utilized. When this figure reaches 85%-90%, storage space is becoming scarce and additional warehouses may be considered. Therefore, when the activity holding an excess reports a storage utilization over, say, 85%, the storage cost figure could include fixed costs.

Two methods of estimating storage cost are suggested. One is to find direct labor costs for warehousing functions from Navy fiscal records, add an overhead percentage, and compare the result with the total value of material in inventory. Another way is to use the fees charged by commercial storage firms considering the size and value of the item. In either case, the estimate must be adjusted according to whether the item utilizes outdoor, indoor, refrigerated, or special handling storage.

A figure of .01 for annual storage cost of ordinary general stores material is often used — i.e., it costs 1 cent per year to store hardware worth 1 dollar. Commercial firms usually use a somewhat higher figure.

Some Hypothetical Values of the Length of the Economic Retention Period

To illustrate the use of the formula, let us suppose that GSSO has an excess stock of an item of pipe fittings located in Norfolk. Suppose the total stock on hand is 10,000 units and the average expected future usage is 100 per year - a stock sufficient for 100 years. Suppose GSSO estimates the disposal return for excess pipe fittings in surplus sales at .15; uses an interest rate of .03; estimates the probability of obsolescence of these fittings at .02 per year; and assumes the storage cost to be .01 per year per dollar value of material. Using these values, we find that $x = 24.65$. This means that it is economical to keep only a stock for a period of 25 years (2,500 units); the remaining 7,500 units should be disposed of.

In this basic example, we have used $D = .15$, $i = .03$, $p = .02$, and $r = .01$. Let us examine a few other sets of values for these variables. It will be seen that a higher disposal value, D, will mean that x is lower - or, in other words, that it is better to dispose of more material. In the case above if the disposal value had been .50 instead of .15, stock for only 11 years would be retained.

If the interest rate had been .06 instead of .03 in the basic example above, the economic retention period would be lowered from 25 years to 18 years.

If the probability of obsolescence is lower, it is of course more desirable to retain stock. Thus, in the basic example if we lower the value of p from .02 to .01, the economic retention period rises from 25 to 29 years.

If the storage costs are raised, it is less desirable to retain material. In the basic example, a storage cost, r, of .02 instead of .01 lowers the economic retention period from 25 to 19 years.

The maximum length of the economic retention period will be found when the disposal value, probability of obsolescence, and storage costs are all low. For example, if D = 0, i = .01, p = 0, and r = .005, it would be economical to retain stock for 110 years.

At the other extreme, if disposal value is very high, storage costs are also high, and yet there is a high probability of obsolescence, the economic retention period will be very short. If D = .75, i = .06, p = .30, and r = .10, stock for only one half of a year would be retained.

For many spare parts with a high probability of obsolescence, a low disposal value is expected. In a typical case, we might find D = .15, i = .03, p = .30, and r = .01. Then, the economic retention period is only five years. If the disposal value should drop to zero, it is better to retain a larger stock, shown in the tables as a stock for 10 years.

4. THE FORMULA AS A PRACTICAL DECISION RULE

There are two major objections to the use of this formula as a practical decision rule by operating activities. Both of these objections should be evaluated by comparison with the present methods of making retention-disposal decisions.

The first objection is that it is too difficult or costly to determine proper values of D, i, p, and r for items or classes of material. This objection was dealt with above when it was pointed out that present methods for retention-disposal decisions consider these same factors in a qualitative "experienced judgment."

The second objection, and a much more important one, is that the assumptions are too restrictive and unrealistic. It is quite true that they are restrictive and that additional factors should be considered in many problems. A discussion of means of relaxing the assumptions and modifying the formula for other factors is therefore in order. However, many if not all the methods now used in making retention-disposal decisions involve assumptions which are equally or even more unrealistic, and often these assumptions are hidden or unrealized. The simple decision rule most commonly used today is probably the choice of a single number of years' stock to be retained for all items, determined by dividing the average value of all items (per pound or cube) by the annual cost of warehousing (per pound or cube). The use of an across-the-board figure of this type certainly contains a set of rather broad assumptions.

Nevertheless, if the formula presented in this report can be modified and generalized, by relaxing the assumptions made earlier, to cover a wider variety of possible cases, it would be a more exact and more powerful tool. It is likely that a general formula covering all probable cases and factors would prove too complex or unwieldy for practical decision-making. However, it might be possible then to find a simpler, useful retention-disposal rule which would retain the validity of the complete general formula in most cases. Pending the achievement of these research objectives, it is suggested that the present formula represents the best choice for a practical decision rule.

Let us examine the assumptions now. The vital assumption is, of course, that future demand is known. It is obvious that the Navy's predictions of future demand for individual items cannot be expected to show great accuracy, since demand is subject to continually shifting military programs and rapid technological changes as well as random or unknown fluctuations due to the inventory practices at various levels in the manufacturer-to-ship supply distribution system. But where we are considering the average demand over a considerable span of future years, the latter fluctuations should be of minor importance. Therefore, if we substitute for the assumption of known annual demand an assumed probability distribution about the average for demand within each year, the effect upon the economic retention period will be relatively small. Moreover, the problems in doing so are rather formidable. On the theoretical side,

it would be necessary to set forth a set of formulas dealing with each type of probability distribution. On the practical side, it would be necessary to collect data on past demand (it can be stated that adequate data for this purpose are not readily available in Navy records) and perform the necessary statistical analyses for each item or class of items, and to assume that these distributions would hold in the future. For the fluctuations of demand in future years due to expected changes in programs or techniques, another estimate of probability would also be necessary. Despite these obstacles and our belief that the length of the economic retention period would not be affected significantly, it is believed that the theoretical problems here are worthy of further research.

A related assumption is that future demand will be constant. In many cases it is more reasonable to postulate a rising or falling future demand. If this trend is assumed to be steady and known, the formula may be modified fairly readily for practical decision-making. In point of fact, however, our real predictions of future demand generally assume neither constancy nor infallibility. Instead we predict various estimated patterns of expected demand with estimated probabilities attached to each. The safety level of stock held results from an estimate that normal or peacetime demand will show a certain variability, not necessarily random. In addition we hold mobilization reserve stocks, and presumably a probability that war will come in any one year could be attached to this expected jump in demand. As before, the practical and theoretical problems in generalizing the formula and using it in practice are great but worthy of study. Meanwhile, it is believed that mobilization reserves and safety level stocks should be set aside or subtracted from stocks on hand prior to employing the present formula, and that the resulting economic retention period will still be the best way to make the retention-disposal decision.

A third assumption is that the new price of the item will remain constant and only three values are to be considered - v, D, and zero. If this assumption is to be replaced, considerable analyses and data would be necessary to justify any substitute. It should be added that it is implicitly assumed that material will be available at time t_x. The possibility that this might not be true because of production bottlenecks has been suggested. Since mobilization reserves are supposed to be equal to requirements until P-day when the nation's productive capacity can meet war requirements, this factor should already have been taken care of.

A fourth assumption that may well be attacked is that the probability of obsolescence and deterioration remains constant in future years. In many cases it would seem more reasonable to postulate a very low chance of obsolescence or deterioration in the first year or so, but an increasing probability as time goes on. As in the case of the third assumption, the refinement could be added at the expense of lengthening the mathematical computations. In view of the rough nature of the estimates used, it would appear to be unnecessary in practical use. In theory, however, obsolescence and deterioration should be treated separately, since deteriorated material should be replaced to meet continuing demands while obsolescence means that demand for the item has dropped to zero. Another factor of importance in practical problems is the existence of a shelf life for an item, e.g., Navy jumpers or aviation starters. In these cases, the shelf life constitutes the maximum technical retention period, and governs where the economic retention period is longer.

It may be added that other adjustments for practical conditions, especially changes in the cost and demand data, can also be made. Thus, a change in expected demand will considerably alter the quantity to be disposed of. If the Navy continues to make a new prediction of future demand for each item every three months and these predictions show the wide variations

observed in the present quarterly reporting system, large stocks of items would move into and out of the disposal category every quarter. To prevent such constant shifting, a reasonable range of limits within which action should be taken may be set. For example, the formula could be applied to each item in excess only once a year. A rule could be adopted that disposal action would be instituted only if the quantity of disposal material determined from the formula were over two years' stock or over $1,000 in value. Special rules could be added to cover cases where past demand drops to zero for a time, or where the insurance nature or long lead time of an item requires additional safeguards. The individual supply demand control point could thus tailor its disposal program to its own needs while relying on the formula as the basic criterion.

Experimental Use of the Formula

Several Naval activities have begun analysis of use of the formula. A brief comparison of the problems encountered may be useful.

At one activity a retention level of fifty years was in use. It had been developed by dividing the average value of a measurement ton of the material concerned by a holding cost per measurement ton per year. Rough estimates of proper values for D, i, and p made by personnel from the activity indicated that the economic retention period determined by the formula would vary from 10 years to 35 years. No adjustment for mobilization reserves was specified in this case.

At another activity the retention of material beyond 3 or 4 years' stock plus mobilization reserves was being questioned. Application of the formula revealed that a much longer economic retention period, and therefore an enlargement of the stratification stock called "economic reserve," was called for. The economic retention period exceeded the shelf life for many items.

Finally, it may be noted that in a recent BuSandA Instruction a pair of general economic break-even points of 100 years for material not requiring represervation and 22 years for material requiring represervation were set forth by a simplified calculation employing the same variables used in the present formula. It is believed that the formula represents a more exact method of using these factors for retention-disposal decisions.

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A METHOD FOR CALCULATING OPTIMAL INVENTORY LEVELS AND DELIVERY TIME

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The following note is concerned with establishing an inventory control policy for items with extremely low demand. In the event that the expected savings in ordering cost that would result from buying in lots is less than the concomitant increase in carrying charges, it is uneconomical to use a lot size formula. In this event, it is appropriate to use a system consisting of placing orders as units are demanded.

Assume that a businessman owns a fleet of taxicabs which he has repaired at his company's central garage. To avoid ambiguity consider a definite part or item, and consider the question of stocking units of this part. It will be assumed that demand for the item is observed at the end of each day, and that the distribution of demand is Poisson. (In point of fact this last assumption is employed merely to simplify computation; conceptually the distribution may be arbitrary.) The ordering policy consists of ordering precisely the number of units of the part demanded when the demand is observed; i.e., at the end of the day. The delivery or pipeline period is assumed to be constant and the Poisson parameter for this period is μ.

The central problem involves two types of costs: the expense of carrying units of the item on shelves, and the loss due to taxicabs awaiting unavailable stock. Stockage objective is defined to be stock on hand and on order. (It is an interesting consequence of the stockage policy that this is constant.)

The basic problem is to set an optimal stockage objective; that is, one which minimizes the attendent costs mentioned above. For the present, let us determine the optimal policy for a given delivery period. The expected loss, $\phi(S, \mu)$, for stock level S with demand following a Poisson distribution with mean μ for the delivery period, is given by equation (1), which is derived in the appendix.

$$\phi(S, \mu) = S\,[C - (\pi + C)\,P(S+1, \mu)] - \mu\,[C - (\pi + C)\,P(S, \mu)] \tag{1}$$

In this equation, C represents the cost of carrying a unit of stock for one day, π represents the daily cost of being out of a part, and $P(S, \mu) = \sum_{k=S}^{\infty} e^{-\mu} \frac{\mu^k}{k!}$; that is, the cumulative Poisson distribution. The optimal choice of S is the level corresponding to the last cumulative

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[1]The basic research for this paper was carried out at the RAND Corporation. The analysis was refined and completed under the auspices of the Sloan Research Fund, School of Industrial Management, M. I. T. and Indiana University.

probability $P(S, \mu)$ not less than $\left(\frac{C}{\pi + C}\right)^2$. That is, the optimal choice of S, called S_μ, is the integer selected so that

$$P(S_\mu, \mu) \geq \frac{C}{\pi + C} > P(S_\mu + 1, \mu)$$

(In the event S_μ can be selected so that the equality holds on the left above, then $S_\mu - 1$ also provides an optimal choice of S.)

This means that for each μ, and for any S,

$$\phi(S_\mu, \mu) \leq \phi(S, \mu)$$

Since the function on the left depends only on μ the notation $\theta(\mu)$ is adopted for it. Clearly

$$\theta(\mu) = \underset{S}{\text{Min}}\ \phi(S, \mu)$$

Figure 1 and Table 1 indicate the behavior of $\theta(\mu)$ for $\pi = 99$ and $C = 1$. A similar table can be constructed for different $\frac{C}{\pi + C}$ ratios. The true form of Figure 1 is shown in the appendix but Figure 1 is a good approximation. The true form is of the "envelope" type familiar to economists.

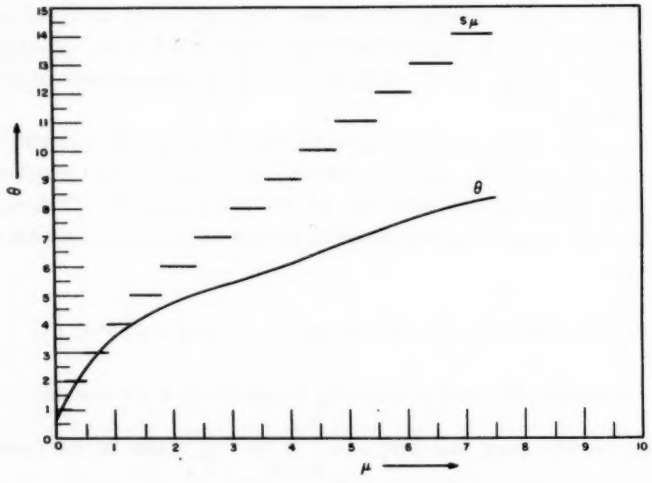


Figure 1 - Relationship between demand rate, μ, optimal stock level $S\mu$ and minimum daily cost $\theta(S\mu)$

[2]This is demonstrated in the appendix, formulas (27) and (28). It is intuitively clear that this is a logical extension of the typical static style goods examples as found in P. Morse and G. Kimball, Methods of Operations Research, New York, 1950, pp. 30-31; T. M. Whitin, The Theory of Inventory Management, Princeton, 1953, pp. 62-72; and K. J. Arrow, T. Harris, and J. Marschak, "Optimal Inventory Policy," Econometrica, July, 1951, p. 258.

TABLE 1
Computation of S_μ and
$\theta(S_\mu)$ for $\pi = 99$, $C = 1$

$\mu = a\lambda$	S_μ	$\theta(S_\mu, \mu)$
0.02	1	.9999
0.15	2	1.9022
0.5	3	2.6938
0.9	4	3.3737
1.3	5	3.9706
1.8	6	4.5266
2.4	7	5.0452
3.0	8	5.5291
3.6	9	5.9799
4.2	10	6.4043
4.8	11	6.8089
5.5	12	7.2022
6.1	13	7.5733
6.8	14	7.9338
7.5	15	8.2830
11.1	20	9.8736
14.9	25	11.2774
19.0	30	12.5690

In the computations, it is convenient to consider μ in terms of two factors: the mean daily demand λ, and the delivery period a. It is then a property of the Poisson distribution that

(2) $$\mu = a\lambda$$

Table 1-a gives values read off Figure 1 and Table 1 for an item with average daily demand λ of .05, $C = 1$, $\pi = 99$, and delivery period $a = 1/2, 1, 2, 2^2, 2^3 \ldots$

TABLE 1-a

(1)	(2)	(3)
$\mu = a\lambda$	S_μ	$\theta(\mu)$
0.025	1	0.99
0.05	1	1.0
0.1	1	1.5
0.2	2	1.9
0.4	2	2.5
0.8	3	3.2
1.6	5	4.3
3.2	8	5.7
6.4	13	7.7

Table 2 is a method of summarizing transportation costs[3] that leads to a simple method of combining transportation costs with $\theta(\mu)$. Tables 1-a and 2 are arranged with the numbers in the left-hand column of each in the same geometric progression.

TABLE 2

(0)'	(1)'	(2)'	(3)'	(4)'
$\mu = a\lambda$	Transport Time a (days)	Average Daily Demand λ	Unit Transportation Cost f(a)	(2)'x(3)' $\lambda[f(a)]$
0.05	1	0.05	200	10
0.1	2	0.05	100	5
0.2	4	0.05	80	4
0.4	8	0.05	40	2
0.8	16	0.05	20	1
1.6	32	0.05	10	0.5
3.2	64	0.05	8	0.4

Table 2 may be positioned vertically next to Table 1-a (slide-rule fashion) in such a manner that the 1 day transport time is lined up horizontally with 0.05 in the first column of Table 1-a. This matches the μ's of Table 2, Column (0)', with the μ's of Table 1-a, Column 1. Table 3 shows Tables 1-a and 2 in the desired position.

TABLE 3

$c = 1$, $\pi = 99$

Table 1-a			Table 2					
(1)	(2)	(3)	(0)'	(1)'	(2)'	(3)'	(4)'	(5)'
$\mu = a\lambda$	S_μ	$\theta(\mu)$	$\mu = a\lambda$	Transport Time a	Average Daily Demand λ	Unit Transportation Cost f(a)	(2)'x(3)' $\lambda[f(a)]$	Total (3) + (3)' $\theta(\mu) + \lambda[f(a)]$
0.025	1	0.99						
0.05	1	1	0.05	1	0.05	200	10	11
0.1	1	1.5	0.1	2	0.05	100	5	6.5
0.2	2	1.9	0.2	4	0.05	80	4	5.9
0.4	2	2.5	0.4	8	0.05	40	2	4.5
0.8	3	3.2	0.8	16	0.05	20	1	4.2
1.6	5	4.3	1.6	32	0.05	10	0.5	4.8
3.2	8	5.7	3.2	64	0.05	8	0.4	6.1
6.4	13	7.7	6.4	128	0.05	8	0.4	8.1

[3]See Appendix 3, the effect of transportation costs.

The right-hand column of Table 3 contains the total daily cost of stocking, running out of stock, and transportation. The lowest figure in this column is 4.2. Columns (2) and (1)' indicate that the initial stock and transportation time that give this minimum figure are 3 units and 16 days respectively.

For a daily demand of 0.025, Table 3 would be changed to Table 3-a as shown below.

TABLE 3-a

$C = 1$, $\pi = 99$

Table 1-a			Table 2-a					
(1)	(2)	(3)	(0)'	(1)'	(2)'	(3)'	(4)'	(5)'
$\mu = a\lambda$	S_μ	$\theta(\mu)$	$\mu = a\lambda$	Transport Time a	Average Daily Demand λ	Unit Transportation Cost f(a)	(2)'x(3)' $\lambda[f(a)]$	Total (3) + (3)' $\theta(\mu) + \lambda[f(a)]$
0.025	1	0.99	0.025	1	0.025	200	5	5.99
0.05	1	1.0	0.05	2	0.025	100	2.5	3.5
0.1	1	1.5	0.1	4	0.025	80	2	3.5
0.2	2	1.9	0.2	8	0.025	40	1	2.9
0.4	2	2.5	0.4	16	0.025	20	0.5	3.0
0.8	3	3.2	0.8	32	0.025	10	0.25	3.45
1.6	5	4.3	1.6	64	0.025	8	0.2	4.5
3.2	8	5.7	3.2	128	0.025	8	0.2	5.9
6.4	13	7.7	6.4					

This table indicates an initial stock of 2 and a transportation time of 8 days. Similar computations for a daily demand of 0.1 give a result of 3 units initial stock and 16 day transportation time, and for a daily demand of 0.8, we obtain optimal values of a 39-unit stockage objective and 32 days delivery time, giving a total daily cost of 22.6.

Next, let us examine the impact of changes in C. For this purpose we examine the ratios $C/(\pi + C) = 1/10$ and $C/(\pi + C) = 1/1000$.[4] As intuition would lead us to expect, the more expensive items ($C/(\pi + C) = 1/10$) should be transported more rapidly than the less expensive, for increasing the stock level is more costly for expensive items. Tables 3-b and 3-c show the behavior of total costs of carrying, running out, and transporting stock for a day for $C/(\pi + C) = 1/10$ and $C/(\pi + C) = 1/1000$ respectively, for a demand of 0.05 per day. For the high-cost items of Table 3-b the optimum delivery time is 4 days with stockage objective 1. For the cheaper items of Table 3-c, a stockage objective of 7 and a delivery time of 32 days are appropriate. Also, it is evident that the decision can make more difference in the case of high-value items.

[4]The corresponding values of C are 11 and 11/111, when π is fixed at 99.

TABLE 3-b

C = 11; π = 99

(1)	(2)	(3)	(0)'	(1)'	(2)'	(3)'	(4)'	(5)'
$\mu = a\lambda$	S_μ	$\theta(\mu)$	$\mu = a\lambda$	Transport Time a	Average Daily Demand λ	Unit Transportation Cost f(a)	(2)'x(3)' λ[f(a)]	Total (3) + (3)' $\theta(\mu) + \lambda$[f(a)]
0.025	0	2.475						
0.05	0	4.950	0.05	1	0.05	200	10	14.950
0.1	0	9.900	0.1	2	0.05	100	5	14.900
0.2	1	10.857	0.2	4	0.05	80	4	14.857
0.4	1	14.333	0.4	8	0.05	40	2	16.333
0.8	2	19.591	0.8	16	0.05	20	1	20.591
1.6	3	27.522	1.6	32	0.05	10	0.5	28.022
3.2	6	38.456	3.2	64	0.05	8	0.4	38.856
6.4	10	52.448	6.4	128	0.05	8	0.4	52.848

TABLE 3-c

C = 11/111, π = 99

(1)	(2)	(3)	(0)'	(1)'	(2)'	(3)'	(4)'	(5)'
$\mu = a\lambda$	S_μ	$\theta(\mu)$	$\mu = a\lambda$	Transport Time a	Average Daily Demand λ	Unit Transportation Cost f(a)	(2)'x(3)' λ[f(a)]	Total (3) + (3)' $\theta(\mu) + \lambda$[f(a)]
.025	1	.126						
.05	2	.195	.05	1	.05	200	10	10.195
.1	2	.204	.1	2	.05	100	5	5.204
.2	3	.283	.2	4	.05	80	4	4.283
.4	3	.341	.4	8	.05	40	2	2.341
.8	5	.437	.8	16	.05	20	1	1.437
1.6	7	.567	1.6	32	.05	10	.5	1.067
3.2	10	.740	3.2	64	.05	8	.4	1.140
6.4	15	1.004	6.4	128	.05	8	.4	1.404

The above illustrations indicate that the choice of the appropriate mode of transportation may well exert an extremely large influence on the stock level. For example, examination of Tables 3-a and 3-b shows that if the delivery time of 4 days (the optimum for items with C = 11, π = 99) was used for the items of Table 3-c, (C = 11/111; π = 99), the stockage objective would be 3 units instead of 7, total costs would be incurred that exceed those incurred with a 32 day transport time and a stockage objective of 7 by a factor of 4. Nevertheless, it is all too often the case that transportation costs are taken as given and inventory levels are optimized under these restrictive conditions, if at all.

Suppose, on the other hand, that the 32 day delivery time (optimal for Table 3-c) was used for the items of Table 3-b. The resulting stockage objective would be 3 instead of 1 and

additional costs of more than \$12 per day would be incurred. Thus it is clear that one can make errors of selecting either too expensive or too cheap a mode of transportation. These errors may be of more consequence than errors in selecting the appropriate stockage objective.

Obviously, the geometric progression for $a\lambda$ with ratio of 2 is very crude. The choice of a ratio slightly over unity would lead to more precise results. Tables can be constructed for various ratios of $C/(\pi + C)$ and various demand rates giving the total costs and the resulting daily costs of transportation, stocking, and running out of stock.

Also, the use of a single transportation time tends to build up the desired stockage objective, since there is no emergency form of delivery available. Consequently, the use of a double or multiple transportation system might lead to a lower overall system cost. This possibility is being investigated.

APPENDIX

1. THE OPTIMAL STOCKAGE OBJECTIVE

A single item is considered and its stock at time $t = 0$ is S units ($S \geq 0$). It is assumed that demands on this item are observed at times $t = 1, 2, 3, \ldots$, and $\underline{x}(t)$ is the random variable of demand at time t. The distribution of $\underline{x}(t)$ is assumed Poisson and given by

$$p(j, \lambda) = e^{-\lambda} \lambda^j / j! \qquad j = 0, 1, 2, \ldots \tag{1}$$

(λ is the mean demand per unit time.)

The ordering policy consists of requisitioning $\underline{x}(t)$ units of the item at time t; that is, the size of the order at t is precisely the size of the demand at t. It is assumed that there is a constant pipeline time a (an integral number), so that $\underline{x}(t)$ units of the item are added to the stock at time $(t + a)$.

It is convenient to consider the random variable

$$\underline{X} = \underline{x}(r + 1) + \ldots + \underline{x}(r + a) \tag{2}$$

which has the Poisson distribution

$$p(j, \mu) = e^{-\mu} (\mu)^j / j! \qquad j = 0, 1, 2, \ldots \tag{3}$$

where $\mu = a\lambda$, the mean demand per pipeline time.

The first object is to develop a formula for $\underline{S}(t)$, the random variable of stock at time t.

If $t \leq a$, then no ordered stocks have arrived by time t, since the first order is placed at time $t = 1$. Hence

$$\underline{S}(t) = S - \sum_{i=1}^{t} \underline{x}(i) \qquad (t \leq a)$$

If $t = r + a$, where $r > 0$, then stock ordered at $t = 1, \ldots, r$ have entered the system. Consequently,

$$\underline{S}(t) = S - \sum_{i=1}^{r+a} \underline{x}(i) + \sum_{i=1}^{r} \underline{x}(i) = S - \sum_{i=r+1}^{r+a} \underline{x}(i)$$

(In both cases the random variable subtracted from S is the number of units of the item "on order" at time t. Hence the "stockage objective" S is simply the sum of stocks on hand and on order.)
Hence

$$\underline{S}(t) = S - \underline{X} \qquad (t > a) \tag{4}$$

The treatment given here is entirely confined to the last form for $\underline{S}(t)$. (After time a the system may be considered to have achieved a stable state.)

It is important to note that $\underline{S}(t)$ is a "bookkeeping" stock and is a random variable which may take on negative values. If $\underline{S}(t)$ is negative, this simply means one is in need of $[-S(t)]$ units (at time t) which are not available, and it is assumed that each of these parts produces a non-operable taxi which imposes a cost of π on the system per unit of time. On the other hand, if $\underline{S}(t)$ is positive, this means that one has in stock (at time t) $\underline{S}(t)$ units, and it is assumed that there is a carrying charge C for each such part per unit of time.

These considerations lead one to examine two other random variables

$$\underline{y}(t) = \begin{cases} 0 \text{ if } \underline{S}(t) \geq 0 \\ -S(t) \text{ if } \underline{S}(t) < 0 \end{cases} \tag{5}$$

(the random variable of demand unable to be filled)

$$\underline{z}(t) = \begin{cases} \underline{S}(t) \text{ if } \underline{S}(t) > 0 \\ 0 \text{ if } \underline{S}(t) \leq 0 \end{cases} \tag{6}$$

(the random variable of stock on the shelves)

To compute the distribution function for $\underline{y}(t)$, observe that by (3), (4), and (5)

$$Pr\{\underline{y}(t) = 0\} = p(0, \mu) + \ldots + p(S, \mu) \tag{7}$$

and for $j = 1, 2, 3, \ldots$

$$Pr\{\underline{y}(t) = j\} = p(S + j, \mu) \tag{8}$$

Similarly, by (3), (4), and (6), the distribution function for $\underline{z}(t)$ is

$$Pr\{\underline{z}(t) = 0\} = P(S, \mu) \tag{9}$$

where $P(S, \mu) = p(S, \mu) + p(S + 1, \mu) + p(S + 2, \mu) + \ldots$
and

$$\text{(10)} \qquad \Pr\{\underline{z}(t) = j\} = \begin{cases} p(S - j, \mu) \text{ for } S \geq 1, j = 1, \ldots, S \\ 0 \text{ for } j > S \geq 0 \end{cases}$$

Now consider the cost picture for n units of time from t up to, but not including, (t + n). At time (t + k) the number of taxis out of commission is precisely $\underline{y}(t + k)$, and each such taxi imposes a cost of π on the system. Hence the total cost created by inoperable taxis during the time period under consideration is a random variable

$$\underline{Y} = \pi \sum_{i=t}^{t+n-1} \underline{y}(i)$$

Similarly, insofar as carrying charges are concerned, these total

$$\underline{Z} = C \sum_{i=t}^{t+n-1} \underline{z}(i)$$

The distribution functions for these random variables will not be computed, since the final results depend only on their expected values. Since

$$E(\underline{Y}) = \pi \sum_{i=t}^{t+n-1} E(\underline{y}(i))$$

$$E(\underline{Z}) = C \sum_{i=t}^{t+n-1} E(\underline{z}(i))$$

we need but to compute $E(\underline{y}(i))$ and $E(\underline{z}(i))$.

Using (7) and (8)

$$\text{(11)} \qquad E(\underline{y}(i)) = \sum_{j=0}^{\infty} j\, p(S + j, \mu)$$

whereas, by (9) and (10)

$$\text{(12)} \qquad E(\underline{z}(i)) = \sum_{j=0}^{S} j\, p(S - j, \mu)$$

Hence the total expected cost

$$\phi(S, \mu, n)$$

of the system for n units of time is given by

$$\phi(S, \mu, n) = n \left[\pi \sum_{j=0}^{\infty} j\, p(S + j, \mu) + C \sum_{j=0}^{S} j\, p(S - j, \mu) \right]$$

The formula shows that $\phi(S, \mu, n) = n\,\phi(S, \mu, 1)$, hence selecting S so that $\phi(S, \mu, 1)$ is a minimum, also provides the optimal stockage objective S for n units of time. Consequently, the discussion will continue under the simplifying and non restrictive assumption that $n = 1$. To simplify the notation

$$\phi(S, \mu)$$

will be employed instead of $\phi(S, \mu, 1)$; explicitly,

$$\phi(S, \mu) = \pi \sum_{j=0}^{\infty} j\,p(S + j, \mu) + C \sum_{j=0}^{S} j\,p(S - j, \mu) \tag{13}$$

In order to deal with the above function, several standard results are recorded without proof.

$$k\,p(k, \mu) = \mu\,p(k - 1, \mu) \tag{14}$$

$$\sum_{j=0}^{S} j\,p(j, \mu) = \mu\,[1 - P(S, \mu)] \tag{15}$$

$$\sum_{j=S+1}^{\infty} j\,p(j, \mu) = \mu\,P(S, \mu) \tag{16}$$

$$\frac{dp(k, \mu)}{d\mu} = p(k - 1, \mu) - p(k, \mu) \tag{17}$$

$$\frac{dP(k, \mu)}{d\mu} = p(k - 1, \mu) \tag{18}$$

Using (11) and (16) one obtains

$$E(\underline{y}) = \sum_{j=0}^{\infty} j\,p(S + j, \mu) = \mu\,P(S, \mu) - SP(S + 1, \mu) \tag{19}$$

whereas, using (12) and (15),

$$E(\underline{z}) = \sum_{j=0}^{S} j\,p(S - j, \mu) = S\,[1 - P(S + 1, \mu)] - \mu\,[1 - P(S, \mu)] \tag{20}$$

Hence using (19) and (20) in (13),

$$\phi(S, \mu) = S\,[C - (\pi + C)\,P(S + 1, \mu)] - \mu\,[C - (\pi + C)\,P(S, \mu)] \tag{21}$$

The object is to determine the integer S so as to minimize $\phi(S, \mu)$. A direct computation using (12) yields

(22) $$\phi(S, \mu) - \phi(S - 1, \mu) = C - (\pi + C) P(S, \mu)$$

(This same formula can be developed directly, and, in fact, more simply by using (13)—the present approach was adopted since it is important to have (21)).

The function $P(S, \mu)$ decreases as S increases. Hence

$\phi(S, \mu) - \phi(S - 1, \mu)$ increases as S increases. This statement has important consequences.

(23) If $\phi(1, \mu) - \phi(0, \mu) > 0$, then the optimal S is 0.

(24) If $\phi(1, \mu) - \phi(0, \mu) = 0$, then there are two optimal S's, 0 and 1.

If $\phi(1, \mu) - \phi(0, \mu) < 0$, then two cases arise:
There is an $S_\mu > 0$ such that

(25) $$\phi(S_\mu, \mu) - \phi(S_\mu - 1, \mu) = 0$$

There is an $S_\mu > 0$ such that

(26) $$\phi(S_\mu, \mu) - \phi(S_\mu - 1, \mu) < 0 < \phi(S_\mu + 1, \mu) - \phi(S_\mu, \mu)$$

In the first case, (25), it is easy to see that $\phi(S, \mu)$ is a minimum at S_μ and $S_\mu - 1$ and nowhere else.

In the second case, (26), $\phi(S, \mu)$ is a minimum at S_μ and nowhere else.

Note that by using (22), the inequality (26) is equivalent to

$$P(S_\mu, \mu) > \frac{C}{\pi + C} > P(S_\mu + 1, \mu)$$

whereas the inequality (23) is equivalent to

$$1 = P(0, \mu) > \frac{C}{\pi + C} > P(1, \mu)$$

Hence, regardless of the subsidiary conditions of (23) and (26),

A minimum for $\phi(S, \mu)$ exists at a unique S if and only if there is an integer S_μ such that

(27) $$P(S_\mu, \mu) > \frac{C}{\pi + C} > P(S_\mu + 1, \mu)$$

and the minimum is taken on at S_μ.

A similar discussion involving (22), (24), and (25) shows that

A minimum for $\phi(S, \mu)$ exists at precisely two values of S if and only if there is an integer S_μ such that

(28) $$P(S_\mu, \mu) = \frac{C}{\pi + C}$$

and the minimum is taken on at $(S_\mu - 1)$ and S_μ.

Moreover, the statements of (27) and (28) cover all the cases. In particular, it should be mentioned that a minimum for $\phi(S, \mu)$ always exists.

It is significant to observe that the optimal stockage objective S_μ depends only on the ratio $\frac{C}{\pi + C}$; while writing (21) in the form

$$\frac{\phi(S_\mu, \mu)}{\pi + C} = S_\mu \left[\frac{C}{\pi + C} - P(S_\mu + 1, \mu)\right] - \mu \left[\frac{C}{\pi + C} - P(S_\mu, \mu)\right]$$

shows that the minimum cost divided by $(\pi + C)$ also depends only on the ratio $\frac{C}{\pi + C}$.

In view of (6) and (20), the expected number of units of the item on the shelves at any time t is

$$E(z) = \sum_0^S j\, p(S - j, \mu)$$

$$= S\,[1 - P(S + 1, \mu)] - \mu\,[1 - P(S, \mu)],$$

if the initial stock is S.

The second form shows that if the stockage objective S is positive, then

(29) $$E(z) < S.$$

If $S = S_\mu$ (that is, an optimal selection of S is made) then (27) and (28) imply

$$E(z) > S_\mu \left[1 - \frac{C}{\pi + C}\right] - \mu \left[1 - \frac{C}{\pi + C}\right]$$

Hence

(30) $$E(z) > (S_\mu - \mu)\left(\frac{\pi}{\pi + C}\right)$$

2. THE EFFECT OF μ

It will be recalled that in (3) μ was defined to be $a\lambda$; that is, the product of pipeline time a and mean demand per unit time λ. It is a considerable convenience to have the optimal S and the minimum $\phi(S, \mu)$ appear as functions of μ alone. For example, this means that if 10 units of an item be the stockage objective where $\lambda = .06$ and $a = 70$, then the same number of another item should be stocked where $\lambda = .12$ and $a = 35$. But the real convenience occurs in examining the effect of pipeline time.

A simple computation employing (18) and (14) shows that, for fixed $S \geq 0$,

(31) $$\phi'(S, \mu) = \frac{\partial \phi(S, \mu)}{\partial \mu} = (\pi + C) P(S, \mu) - C$$

In particular

$$\phi'(0, \mu) = (\pi + C) - C = \pi$$

It is convenient to let

$$S = k$$

In view of the fact that $P(k, \mu)$ increases strictly as μ increases ($k > 0$) and $P(k, 0+) = 0$, while $P(k, \infty) = 1$, formula (31) shows that:

(32) $\phi'(k, \mu)$ increases strictly from $-C$ to π as μ increases from 0 to ∞, if $k > 0$

(33) $\phi'(0, \mu) = \pi$

For $k > 0$, define

$$\mu_k$$

to be the unique solution of

$$\phi'(k, \mu) = 0,$$

and let

$$\mu_0 = 0.$$

Hence

(34) $$\phi'(k, \mu_k) = \begin{cases} 0 & \text{if } k > 0 \\ \pi & \text{if } k = 0 \end{cases}$$

It will be shown that:

(35) $$\mu_k < \mu_{k+1}, \quad k = 0, 1, 2, \cdots.$$

For any $\mu > 0$

$$P(k + 1, \mu) < P(k, \mu)$$

and hence by (31),

$$\phi'(k + 1, \mu) < \phi'(k, \mu_k) = 0$$

Consequently, by (32), the solution of $\phi'(k+1, \mu) = 0$ is greater than μ_k. But by definition, this solution is μ_{k+1}, and hence (35) is proved.

Define

$$\theta(\mu) = \min_S \phi(S, \mu) \tag{36}$$

(Note that θ is the function giving minimum cost.)

The properties of θ are to be investigated. To this end, consider any fixed $k \geq 0$ and let μ be any number such that

$$\mu_k < \mu < \mu_{k+1} \tag{37}$$

Because of (32), (33), and (34)

$$0 < \pi = \phi'(0, \mu_0) = \phi'(0, \mu)$$

$$0 = \phi'(k, \mu_k) < \phi'(k, \mu) \text{ for } k > 0$$

$$0 = \phi'(k+1, \mu_{k+1}) > \phi'(k+1, \mu)$$

Hence, for $k \geq 0$,

$$\phi'(k, \mu) > 0 > \phi'(k+1, \mu),$$

and using (31) an easy computation shows that

$$P(k, \mu) > \frac{C}{\pi + C} > P(k+1, \mu).$$

Hence by (27) and (36)

$$\theta(\mu) = \phi(k, \mu). \tag{38}$$

If $k \geq 0$ and

$$\mu = \mu_{k+1}, \tag{39}$$

then by (31)

$$0 = \phi'(k+1, \mu_{k+1}) = (\pi + C)\, P(k+1, \mu_{k+1}) - C$$

Hence

$$P(k+1, \mu_{k+1}) = \frac{C}{\pi + C}$$

and by (28) and (36)

(40) $$\theta(\mu) = \phi(k+1, \mu) = \phi(k, \mu).$$

Finally, one needs to know the value of $\theta(\mu)$ for $\mu = \mu_0 = 0$. A direct computation using (21) and (36) shows that

(41) $$\theta(0) = 0 = \phi(0, 0)$$

Drawing together the facts in (37) to (41), one obtains a complete determination of $\theta(\mu)$; namely,

(42) $$\theta(\mu) = \phi(k, \mu) \text{ for } \mu_k \leq \mu \leq \mu_{k+1},\ k \geq 0.$$

(Formula (40) guarantees that θ is single valued at each of the points μ_k, $k > 0$.)

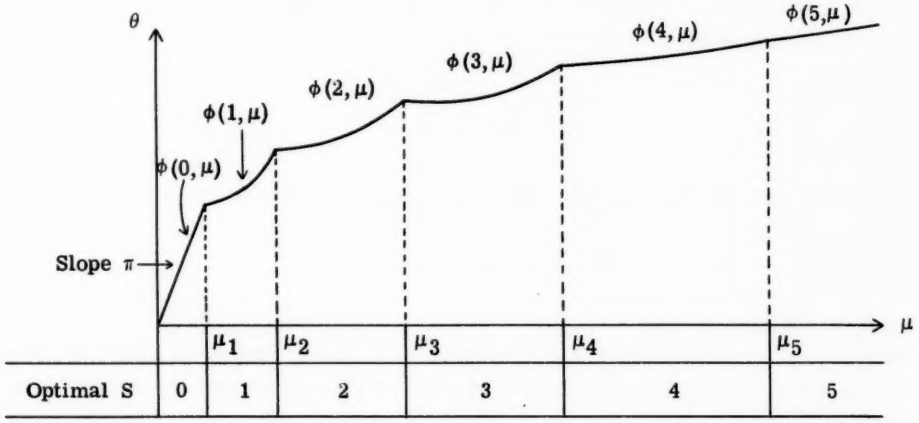
The basic properties of the function θ are now easy to determine.

1° $\theta(\mu)$ is continuous for $\mu \geq 0$

2° In the interval $0 \leq \mu \leq \mu_1$, $\theta(\mu)$ is a straight line through the origin with slope π (see (34) and (42)).

3° In the interval $\mu_k \leq \mu \leq \mu_{k+1}$, $k > 0$, $\theta(\mu)$ is concave upward (see (32) and (42)). (In fact, the slope at μ_k from the right is zero and the slope at μ_{k+1} from the left is positive.

4° θ is strictly increasing as μ increases. Consequently, the complete graph of θ is of the following form



(The graph for $\theta(\mu)$ drawn in the body of the paper—see Figure 1—does not appear to have this form because $\pi = 99$, a large initial slope, while $\mu_1 = .02$, $\mu_2 = .15$, $\mu_3 = .5$, etc.,—see Table 1—and a smooth curve has been fitted through the θ's at these points.)

3. THE EFFECT OF TRANSPORTATION COSTS

Assume that with a pipeline time of a, each unit of the item is transported at a cost

(43) f (a)

It is reasonable to suppose that

$$\infty = f(0) > f(1) > f(2) > \ldots \longrightarrow \epsilon > 0 \tag{44}$$

As in part I of the Appendix, assume that the mean demand per unit time is λ, and recall that $\mu = a\lambda$.

Following the line of discussion initiated after (10), consider the time interval from t up to, but not including, t + n. The units of the item ordered at times (t - a), . . . , (t - a + n - 1) will enter the system at times t, . . . , (t + n - 1) respectively and have transportation charges attached. The total number of these units is a random variable,

$$\underline{x}(t - a) + \ldots + \underline{x}(t - a + n - 1)$$

since the ordering policy calls for requisitioning $\underline{x}(t)$ units at time t. The expected value of this random variable is

$$n\lambda$$

(See (1) and the discussion on (2).)
Consequently, transportation costs for n units of time have an expected value

$$n\lambda\, f(a),$$

and as with ϕ, the fact that n appears as a factor permits a continuation of the discussion under the simplifying assumption that n = 1.

Let

$$T(S, a, \lambda)$$

be the total cost per unit time—transportation costs included. Then since $\mu = a\lambda$

$$T(S, a, \lambda) = \phi(S, a\lambda) + \lambda f(a) \tag{45}$$

where $\phi(S, a\lambda)$ is given by (19).

The problem is to determine S and a so as to minimize T. The problem is simplified by the fact that S does not appear in the portion of formula (44) attributable to transportation costs, namely $\lambda f(a)$.

Note that λ is fixed and a is an integer. Moreover

$$\underset{S}{\text{Min}}\ \phi(S, a\lambda) = \theta(a\lambda)$$

for each a = 0, 1, 2, . . . and θ is completely described at the end of part II. Hence to obtain $\underset{a,\,S}{\text{Min}}\ T(S, a, \lambda)$, one need but examine the sequence of numbers

$$\theta(a\lambda) + \lambda f(a),\ a = 0, 1, 2 \ldots$$

and select the smallest, if any. In fact, the numbers

$$\theta(0),\ \theta(\lambda),\ \theta(2\lambda),\ \theta(3\lambda)\ \ldots$$

may be read directly off a graph of the type found at the end of part II. Since these numbers are strictly increasing, there will generally be a minimum for some $a = a_0$.

Once the optimal a_0 has been determined, the optimal stockage objective S is given by the interval in which $a_0\lambda$ falls for if

$$\mu_k < a_0\lambda \leq \mu_{k+1}$$

then the optimal S is k.

An easy manner of doing the computation (using a geometric progression of a's) is explained in the body of the paper.

* * *

A DYNAMIC TRANSPORTATION MODEL[1]

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1. INTRODUCTION

I. Heller [1] has given a linear programming formulation of a discrete variable transportation problem which arose in connection with routing Navy tankers. The problem differs from the classical Hitchcock-Koopmans problem treated by Dantzig [2] and others, in that there are two sets of port requirements to be satisfied. One set is associated with a short period of time while the other set is associated with a longer period containing the first. Besides the spatial requirements of the short period, there are only specified times within that period when an activity may be performed. Heller also proposed a method to solve the problem, but the algorithm he outlines appears to be computationally infeasible for most problems.

In this note we consider an extension of Heller's formulation, and propose and justify a method to solve the problem. We show that the timing restrictions of the problem may be included in the linear programming model and that its optimum solution is the same as that of a certain Hitchcock-Koopmans problem. The method of [2] may thus be used to obtain minimum solutions.

2. STATEMENT AND FORMULATION OF THE PROBLEM

During the year a homogeneous product (in loads as units) is to be transported in amounts $A_1, \ldots, A_m$ respectively, from each of m lifting ports and received in amounts $B_1, \ldots, B_n$ respectively, at each of n destination ports. We assume a single type of transportation, say ships, and that $\sum_i A_i = \sum_j B_j$. The ballast (empty) travel cost, c_{ij}, and steaming time, t_{ij}, between ports j and i are known for all combinations (i, j).

A specified part of the year's shipping program is to be accomplished during the first month of the year. That is, quantities a_i are to be shipped from port i during the first month. Associated with each cargo a lifting date and destination port are specified. There are b_j ships located at destination port j which can reach some lifting port in time to make a lift during the month. Associated with each ship is a date on which it first becomes available.

How should the available ships be scheduled to perform the specified months shipping program such that the requirements of the month and year are satisfied, and such that the expected ballast travel cost over the whole year is a minimum?

Starting with Heller's formulation, the problem takes the following form: Determine integers x_{ij} and y_{ij} such that

$$(1) \qquad f = \sum_{j=1}^{n} \sum_{i=1}^{m} (x_{ij} + y_{ij})\, c_{ij}$$

is a minimum subject to

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[1]This work was supported by the Office of Naval Research.
[2]Now with Sperry Rand Corporation.

$$\sum_{j=1}^{n} y_{ij} \geq a_i, \quad 1 \leq i \leq m \tag{2}$$

$$\sum_{i=1}^{m} y_{ij} \leq b_j, \quad 1 \leq j \leq n \tag{3}$$

$$\sum_{j=1}^{n} (x_{ij} + y_{ij}) = A_i, \quad 1 \leq i \leq m \tag{4}$$

$$\sum_{i=1}^{m} (x_{ij} + y_{ij}) = B_j, \quad 1 \leq j \leq n \tag{5}$$

$$x_{ij} \geq 0, \quad y_{ij} \geq 0, \quad \text{for all } i, j. \tag{6}$$

The x_{ij} and y_{ij} represent the number of ships sent in ballast from j to i during the first and remaining eleven months of the year, respectively. Inequality (2) says, in effect, that the number of ships which leave a port during the first month of the year cannot be less than the specified number of lifts which must be made at that port during the same period. Inequality (3) says we cannot use more ships than are available. Equations (4) and (5) state the conditions on the whole years program.

Constraint (2) immediately gives rise to a difficulty, viz that the optimal solution may require a destination to receive more loads than it has storage capacity to handle. This situation may be prevented by introducing further constraints at the expense of complicating the problem. Here we wish to avoid these complications by tolerating the possibility of obtaining somewhat larger minimum solutions. This we do by restricting constraint (2) to be an equation, i.e.,

$$\sum_{j=1}^{n} y_{ij} = a_i, \quad 1 \leq i \leq m. \tag{7}$$

In other words, we must send no more loads to a destination port than are required by that port.

Further, a solution to (3)-(7) does not necessarily represent a feasible schedule, i.e., a solution which satisfies all conditions of the problem. It is not unlikely that the solution requires a ship to make a lift which it cannot possibly make at the specified time. There is also no guarantee that a feasible schedule even exists. The fact that the number of available ships is not less than the number of lifts is not sufficient for feasibility.

We make the assumption that there is only one lift to be made at each lifting port and only one ship available at each destination port. If such is not the case there is no loss in considering that each cargo and each ship is located at a distinct port. As a consequence $a_i = b_j = 1$ for all i, j.

Now, in order for a ship located at j to be able to make a lift at i, it is necessary that

$$t_{ij} \leq r_i - d_j ,$$

where r_i is the time when the ith lift is to be made and d_j is the time when the jth ship is available.

We define an m by n deployment matrix, (δ_{ij}), such that

$$\delta_{ij} = \begin{cases} 1, & \text{if } t_{ij} \leq r_i - d_j , \\ 0, & \text{if } t_{ij} > r_i - d_j . \end{cases} \tag{8}$$

Each column of this matrix represents a ship located at a specified port at a specified time. Each row represents a cargo with similar location and time restrictions. A one (zero) in the intersection of a row and column means that the associated ship can (cannot) make the corresponding lift. Thus, the matrix contains all possibilities for routing ships during the month.

Clearly, it is necessary that (δ_{ij}) contain at least m non-null columns and rows, respectively, in order for a feasible schedule to exist. Further, it is not difficult to see that if there exists an m by m permutation matrix, P, such that for some m by m submatrix, R, of (δ_{ij})

$$\text{trace } (PR) = m ,$$

then we have a necessary and sufficient condition for the existence of a feasible schedule. There are fairly simple methods for determining the existence a permutation P and thus, from the outset, the existence of feasible schedules may be determined.

We now rewrite (1) as

$$f = \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} x_{ij} + s_{ij} y_{ij}) \tag{9}$$

where

$$s_{ij} = \begin{cases} c_{ij}, & \text{if } \delta_{ij} = 1 , \\ \omega, & \text{if } \delta_{ij} = 0 . \end{cases}$$

The quantity ω is an arbitrary positive number so large that any y_{ij} associated with it cannot appear with positive value in any minimal solution to the problem. This device excludes the possibility of obtaining non-feasible schedules. If f_0 is the minimal value of (9) for the class of feasible solutions, then it is sufficient to select ωf_0.

The problem now becomes: Determine integers x_{ij} and y_{ij} such that (9) is a minimum subject to (3)-(7).

It is to be remarked that the possibility that a single ship may make two or more lifts during the first month is not excluded from the above formulation. In constructing (δ_{ij}) for a particular problem, all ships which make lifts before the first month and which become available before or during the first month of the year are considered first. If, now, the problem

requirements indicate that a lift is to be initiated and completed during the first month, then we add another column to the deployment matrix, in accordance with (8), to represent the ship that becomes available from the completed lift. Which ship this column actually represents may be determined from the minimal solution.

3. SOLUTION OF THE PROBLEM

We now consider a transformation of constraints (3)-(7). It is shown that, under this transformation, the problem is equivalent to a subsystem of a Hitchcock-Koopmans transportation problem. In this form integer solutions may easily be obtained by the method of [2].

Let

$$x_{ij} = x_{ij}^{(1)} + x_{ij}^{(2)} \tag{10}$$

for all i, j, where $x_{ij}^{(1)}$ and $x_{ij}^{(2)}$ are non-negative. Substituting (10) into (4) and (5), and subtracting (7) from (4) and (3) from (5), respectively, obtains

$$\sum_{j=1}^{n} (x_{ij}^{(1)} + x_{ij}^{(2)}) = A_i - a_i, \quad 1 \le i \le m \tag{11}$$

$$\sum_{i=1}^{m} (x_{ij}^{(1)} + x_{ij}^{(2)}) - p_j = B_j - b_j, \quad 1 \le j \le n \tag{12}$$

where the p_j are non-negative slack variables included to make the results equations.

From the statement of the problem, the right-hand sides of (11) and (12) are non-negative. Thus, we may write

$$p_j = \sum_{i=1}^{m} x_{ij}^{(2)} \tag{13}$$

Substituting (13) into (12) obtains

$$\sum_{i=1}^{m} x_{ij}^{(1)} = B_j - b_j, \quad 1 \le j \le n. \tag{14}$$

Further, it is clear that (3) may be written

$$\sum_{i=1}^{m} (y_{ij} + x_{ij}^{(2)}) = b_j, \quad 1 \le j \le n. \tag{15}$$

Now consider the problem of minimizing (1) subject to (7), (10), (11), (14) and (15). Clearly, every solution to this problem yields a solution of (3)-(7) having the same value of f in (1). To prove the converse, let

$$x_{ij}^{(1)} = (B_j - b_j)\, x_{ij} \Big/ \sum_{k=1}^{m} x_{kj}$$

and

$$x_{ij}^{(2)} = (b_j - \sum_{k=1}^{m} y_{kj})\, x_{ij} \Big/ \sum_{k=1}^{m} x_{kj}$$

for $1 \le j \le n$. It is easily verified that these values of $x_{ij}^{(1)}$ and $x_{ij}^{(2)}$ satisfy the constraints of the transformed problem.

Finally, that the transformed problem is a subsystem of a Hitchcock-Koopmans problem is obvious. Specifically, consider the problem of minimizing

$$F = \sum_{p=1}^{2m} \sum_{q=1}^{2n} d_{pq}\, U_{pq}$$

subject to

$$\sum_{q=1}^{2n} U_{pq} = H_p \qquad 1 \le p \le 2m$$

$$\sum_{p=1}^{2m} U_{pq} = K_q \qquad 1 \le q \le 2n$$

$$U_{pq} \ge 0 .$$

Then we obtain the following relations:

$$y_{ij} = U_{ij} \qquad A_i - a_i = H_{i+m}$$

$$x_{ij}^{(1)} = U_{i+m,\, j+n} \qquad b_j = K_j$$

$$x_{ij}^{(2)} = U_{i+m,\, j} \qquad B_j - b_j = K_{j+n}$$

$$a_i = H_i$$

$$d_{pq} = \begin{cases} c_{ij}, & \text{when } U_{pq} \text{ is an } x_{ij}^{(1)},\ x_{ij}^{(2)} \text{ or } y_{ij} \\ \omega, & \text{otherwise.} \end{cases}$$

The slack variables, p_j, represent those ships which become available at j for lifts during the first month of the year but are not included in the months schedule. It is assumed that they are sent to a dummy lifting port at zero cost though, in reality, they remain at j to be considered for lifts during some subsequent month of the year.

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ANALYTIC STUDY OF WAR GAMES[1]

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The author describes the results of a study of the mathematical aspects of a proposed war game model.

INTRODUCTION

This paper presents the results of a mathematical investigation of the model for a war game that has been proposed by Brig. Gen. L. I. Davis, USAF. The model is a deterministic model, the evolution of the war being governed by a set of strategic parameters and the initial values of the strategic variables involved. These variables are thought of as the strengths of the two opponents offensive and defensive forces-in-being and the rates of their productions which supply these forces. So long as the strategic parameters are held fixed throughout one "play" of the war it is not quite correct to speak, technically, of a game; only when these parameters are permitted to be changed during the course of a play the war becomes properly a game of strategy.

In the present investigation the set of strategic parameters is considered fixed throughout one play of the war. It was felt that this restrictive assumption, while certainly unnecessary in any numerical investigation, was needed here so as to make the model somewhat more easily amenable to analytic study. The basic problem is then the determination of these parameters such that, for a given set of initial values of the strategic variables, victory is guaranteed for one party. This raises the further question as to what conditions shall constitute a victory (or a defeat). Although the concept of victory is clearly understood intuitively, it is here necessary to define it precisely. Perhaps the most natural way is to describe victory in terms of the magnitudes of the strategic variables and thus to define as victor that party whose offensive and defensive forces-in-being as well as whose rate of production are ultimately, i.e., when time increases indefinitely, larger than those of its opponent. Despite the drawback of the ultimacy statement which focusses attention only at the eventual outcome but disregards entirely prior happenings, this is the definition that is used throughout here.

The present paper is divided into two parts. In Part I we state the various results obtained with a minimum of mathematical details and discuss their significance. Part II contains the mathematical techniques used to derive these results. Under Remarks, we make a number of suggestions as to how the present model might be modified and point up some of the mathematical difficulties that are likely to arise from such modifications.

[1]This work was done under contract with the Armament Branch, Air Research and Development Command, USAF.

PART I

1. THE MODEL

Let A and B denote two opposing parties and designate by

x (u) the strength of A's (B's) offensive force-in-being,

y (v) the rate of A's (B's) production,

z (w) the strength of A's (B's) defensive force-in-being.

The model of Brig. Gen. L. I. Davis as illustrated by Figure 1 relates these strategic variables to one another under the assumption that attrition suffered by a force due to attack upon it is of Lanchester type. In this connection it is convenient to think of the rates of production also as forces-in-being.

The complete mathematical description of the model is given by the set of six non-linear ordinary differential equations of the first order stated below. As a guide to their interpretation, consider (for example) the first. The total attrition of A's offensive force-in-being, $x(t)$, is the sum of the attrition $k_{4A} x\, w$ due to B's offensive force-in-being, $w(t)$, and the attrition $k_{1A} x \{b u - j_A z\}_+$ due to the positive excess of B's offensive force-in-being, $b\, u(t)$, that is attacking A's offensive force-in-being, over that part of A's defensive force-in-being, $j_A z(t)$, which is allocated to defend A's offensive force-in-being. The constants k_{1A}, k_{4A} are measures of the effectiveness of the particular type of force involved. Taking into account the fraction of A's rate of production, $k_A y(t)$, that is allocated to supply A's offensive force-in-being, one can write down the differential equation

$$\frac{dx}{dt} = k_A y - k_{1A} x \{b u - j_A z\}_+ - k_{4A} x w$$

which defines the rate of change of the strength of A's offensive force-in-being, $x(t)$, throughout the course of the war.

Thus we find

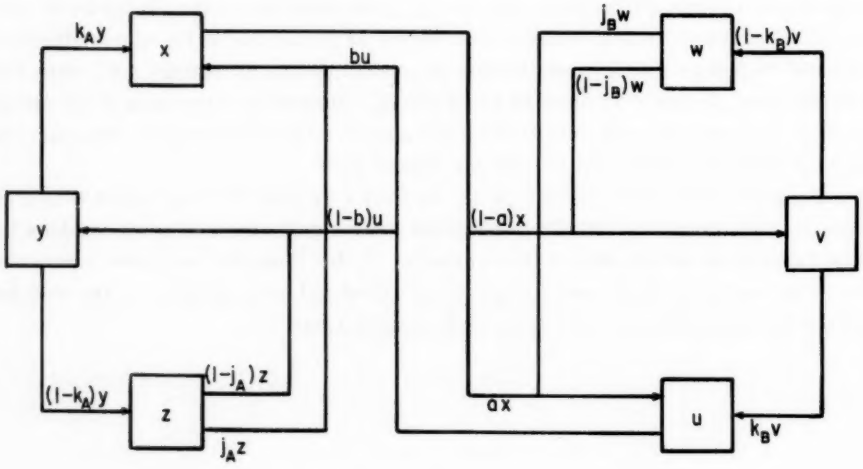


Figure 1 - War "A" vs "B"

$$\frac{dx}{dt} = k_A y - k_{1A} x \{bu - j_A z\}_+ - k_{4A} x w$$

$$\frac{dy}{dt} = \frac{1}{\tau}(P_c - y) - k_{3A} y \{(1-b)u - (1-j_A)z\}_+$$

$$\frac{dz}{dt} = (1-k_A) y - k_{2A} z u$$

(1.1)

$$\frac{du}{dt} = k_B v - k_{1B} u \{ax - j_B w\}_+ - k_{4B} u z$$

$$\frac{dv}{dt} = \frac{1}{\tau}(P_c - v) - k_{3B} v \{(1-a)x - (1-j_B)w\}_+$$

$$\frac{dw}{dt} = (1-k_B) v - k_{2B} w x$$

in which the effectiveness constants are assumed to be

$$k_{1A} = \frac{1}{\tau_{AB} u(0)}, \qquad k_{1B} = \frac{1}{\tau_{BA} x(0)},$$

$$k_{2A} = \frac{1}{\tau_{AD} u(0)}[(1-j_A)(1-b) + j_A b], \qquad k_{2B} = \frac{1}{\tau_{BD} x(0)}[(1-j_B)(1-a) + j_B a],$$

(1.2)

$$k_{3A} = \frac{1}{\tau_{AP} u(0)}, \qquad k_{3B} = \frac{1}{\tau_{BP} x(0)},$$

$$k_{4A} = \frac{1}{\tau_{DA} w(0)}[(1-j_B)(1-a) + j_B a], \qquad k_{4B} = \frac{1}{\tau_{DB} z(0)}[(1-j_A)(1-b) + j_A b].$$

The constants figuring in this set are defined as follows:

a (b)	fraction of A's (B's) offensive force-in-being sent against B's (A's) offensive force-in-being
1-a (1-b)	fraction of A's (B's) offensive force-in-being sent against B's (A's) production
j_A (j_B)	fraction of A's (B's) defensive force-in-being allocated to defend A's (B's) offensive forces
k_A (k_B)	fraction of A's (B's) production allocated to supply A's (B's) offensive force-in-being
$1-k_A$ ($1-k_B$)	fraction of A's (B's) production allocated to supply A's (B's) defensive force-in-being
P_c	goal of both A's and B's rates of production

τ time in which the difference between P_c and the initial rate of production of A(B) could be reduced by 1/e*)

τ_{AB} (τ_{BA}) time in which B's (A's) initial offensive force-in-being could reduce A's (B's) offensive force-in-being by 1/e

τ_{AD} (τ_{BD}) time in which attrition in opposing B's (A's) initial offensive force-in-being could reduce A's (B's) defensive force-in-being by 1/e

τ_{AP} (τ_{BP}) time in which B's (A's) initial offensive force-in-being could reduce A's (B's) rate of production by 1/e

τ_{DA} (τ_{DB}) time in which attrition in attacking B's (A's) initial defensive force-in-being could reduce A's (B's) offensive force-in-being by 1/e.

From these definitions it appears that there are 18 strategic parameters entering into the differential equations (1.1). However, it is reasonable to assume that the time constants $\tau_{AB}, \ldots$ which figure in (1.2) are the same for either party; that is,

$$\tau_{AB} = \tau_{BA}, \quad \tau_{AD} = \tau_{BD}, \quad \tau_{AP} = \tau_{BP}, \quad \tau_{DA} = \tau_{DB}. \tag{1.3}$$

Moreover, one can suppose that the time for attrition in opposing B's initial offensive force-in-being to reduce A's defensive force-in-being by 1/e is equal to the time in which attrition in attacking B's initial defensive force-in-being would reduce A's offensive force-in-being by the same amount 1/e; namely

$$\tau_{AD} = \tau_{DA} \tag{1.4}$$

whence, by the second and fourth equation of (1.3)

$$\tau_{AD} = \tau_{DA} = \tau_{DB} = \tau_{BD}. \tag{1.5}$$

These assumptions cut the number of strategic parameters down to 11.

Preliminary numerical investigations, done prior to the present work, were carried out on the National Bureau of Standards automatic computer SEAC under still more restrictive assumptions. Then it was supposed that the initial values of the strategic variables were respectively the same for either party. In view of the definitions (1.2) and the previous assumptions (1.3) and (1.5) this implies that

$$k_{1A} = k_{1B}, \quad k_{3A} = k_{3B},$$

$$\frac{k_{2A}}{k_{2B}} = \frac{k_{4B}}{k_{4A}} = \frac{(1 - j_A)(1 - b) + j_A b}{(1 - j_B)(1 - a) + j_B a}. \tag{1.6}$$

The results of these numerical investigations pointed up, among other things, the relatively insignificant role that is played by the defensive forces in this model. It appeared that

*)Here and in what follows e = 2.71828..., hence 1/e = 0.367879...

defensive forces would hold off attacking forces but at the same time permit the attacking forces to return the next day when the defensive forces would be weaker. Since the purpose of the present study is not to modify the model so as to achieve more realism but rather to investigate the model as proposed, it was felt that no appreciable loss of generality would be incurred if defensive forces were disregarded in our investigation.

2. THE MODEL WITHOUT DEFENSIVE FORCES

For convenience of later reference, let us state the set of equations to which the system (1.1) reduces when the defensive forces are excluded from it. We find

$$\begin{aligned}
\frac{dx}{dt} &= y - bk_{1A}\,xu \\
\frac{dy}{dt} &= \frac{1}{\tau}(P_c - y) - (1-b)\,k_{3A}\,yu \\
\frac{du}{dt} &= v - ak_{1B}\,ux \\
\frac{dv}{dt} &= \frac{1}{\tau}(P_c - v) - (1-a)\,k_{3B}\,vx
\end{aligned} \tag{2.1}$$

where the notations are as before and

$$\begin{aligned}
k_{1A} &= \frac{1}{\tau_{AB}\,u(0)}, \qquad k_{1B} = \frac{1}{\tau_{BA}\,x(0)}, \\
k_{3A} &= \frac{1}{\tau_{AP}\,u(0)}, \qquad k_{3B} = \frac{1}{\tau_{BP}\,x(0)};
\end{aligned} \tag{2.2}$$

the remaining constants a, b, P_c, τ are defined as originally.

We shall find it advantageous to consider the constants $k_{1A}, k_{1B}, k_{3A}, k_{3B}$ rather than the time constants $\tau_{AB}, \tau_{AP}, \tau_{BA}, \tau_{BP}$ as strategic parameters to be changed from one play of the war to the next. Moreover, we feel that it is more reasonable to suppose

$$\frac{\tau_{AB}}{\tau_{BA}} = \frac{\tau_{AP}}{\tau_{BP}} = \frac{x(0)}{u(0)} \tag{2.3}$$

than to make assumptions (1.3), (1.4), (1.5). These relations imply

$$\begin{aligned}
k_{1A} &= k_{1B} \equiv k_1, \\
k_{3A} &= k_{3B} \equiv k_3.
\end{aligned} \tag{2.4}$$

Then the four equations (2.1) may be rewritten as

$$\frac{dx}{dt} = y - bk_1 xu$$

$$\frac{dy}{dt} = \frac{1}{\tau}(P_c - y) - (1 - b) k_3 yu$$

(2.5)

$$\frac{du}{dt} = v - ak_1 ux$$

$$\frac{dv}{dt} = \frac{1}{\tau}(P_c - v) - (1 - a) k_3 vx .$$

They now contain only six strategic parameters which together with the initial values of the four strategic variables will completely determine the outcome of a play of the war.

Note that our assumptions are more general than those that were made previously in the numerical investigations; for equality of the initial values of the strategic variables is no longer required.

Let us now state our definition of victory (or defeat) which shall be the basis of what follows.

DEFINITION. A defeats B if either

(2.6) $$x(t) > u(t),\ y(t) \geq v(t)$$

or

(2.7) $$x(t) \geq u(t),\ y(t) > v(t)$$

for any time $t \geq T$ where T is some sufficiently large positive constant.

3. OFFENSIVE FORCES AGAINST OFFENSIVE FORCES

We begin with the discussion of the relatively simple case when either party throws its entire offensive force-in-being against its opponents' offensive force-in-being, thus leaving either party's production free to build up to their common goal.

In this case the system (2.5) governing the course of the war reduces, by virtue of the fact that $a = b = 1$, to the four equations

$$\frac{dx}{dt} = y - k_1 xu, \quad \frac{dy}{dt} = \frac{1}{\tau}(P_c - y),$$

$$\frac{du}{dt} = v - k_1 ux, \quad \frac{dv}{dt} = \frac{1}{\tau}(P_c - v),$$

of which the second and fourth equations are immediately solvable explicitly. Because of the simplicity of the present situation it is natural to presume that one's intuition about the outcome of the war, namely that the war will be won by that party which starts out with a larger offensive force-in-being and a bigger rate of production, will be proven correct by the results. This will indeed be the case.

Our results are as follows.

LEMMA 1: The rates of production are represented for any time $t \geq t_0 \geq 0$ by

$$y(t) = P_c - (P_c - y_0) \exp\left[-\frac{1}{\tau}(t - t_0)\right],$$

$$v(t) = P_c - (P_c - v_0) \exp\left[-\frac{1}{\tau}(t - t_0)\right],$$

where y_0, v_0 are the initial values taken at time $t = t_0$. Hence $y(t)$, $v(t)$ are monotonically increasing with time and

$$\lim_{t \to \infty} y(t) = \lim_{t \to \infty} v(t) = P_c .$$

COROLLARY 1: The difference $y(t) - v(t)$ between the rates of production is, as time increases, monotonically increasing or decreasing according as $y_0 < v_0$ or $y_0 > v_0$. Hence, if $y_0 > v_0$ then for any time $t \geq t_0 \geq 0$

$$y(t) \geq v(t) .$$

LEMMA 2: The difference $x(t) - u(t)$ between the strengths of the offensive forces-in-being is represented for any time $t \geq t_0 \geq 0$ by

$$x(t) - u(t) = x_0 - u_0 + \tau(y_0 - v_0)\left[1 - \exp\left(-\frac{1}{\tau}(t - t_0)\right)\right]$$

where x_0, y_0, u_0, v_0 are the initial values taken at time $t = t_0$.

COROLLARY 2: The difference $x(t) - u(t)$ between the strengths of the offensive forces-in-being is, as time increases, monotonically increasing or decreasing according as $y_0 > v_0$ or $y_0 < v_0$. Hence, if $y_0 > v_0$ and $x_0 > u_0$ then for any time $t \geq t_0 \geq 0$

$$x(t) > u(t) .$$

THEOREM 1: If, at time $t = t_0 \geq 0$, the strengths of the offensive forces-in-being and the rates of production satisfy

$$x_0 > u_0, \quad y_0 > v_0$$

then for any time $t \geq t_0$

$$x(t) > u(t), \quad y(t) \geq v(t) .$$

Thus, A defeats B.

Although this theorem states sufficient conditions for victory, it does not yield the ultimate values to which the strengths of the offensive forces-in-being tend as time increases indefinitely. These values are obtained in the following lemma.

LEMMA 3: The strengths of the offensive forces-in-being are represented for any time $t \geq t_o \geq 0$ by

$$x(t) = \frac{1}{2k_1}\left(\sqrt{\alpha^2 + 4k_1 P_c} + \alpha\right) - \frac{8W_o^2}{k_1}(t - t_o)\, e^{-\lambda(t-t_o)} +$$

$$\frac{\beta}{2k_1} e^{-\frac{1}{\tau}(t-t_o)} + \frac{1}{k_1} W_1(t) + 0\{\phi(t)\},$$

$$u(t) = \frac{1}{2k_1}\left(\sqrt{\alpha^2 + 4k_1 P_c} - \alpha\right) - \frac{8W_o^2}{k_1}(t - t_o)\, e^{-\lambda(t-t_o)} -$$

$$\frac{\beta}{2k_1} e^{-\frac{1}{\tau}(t-t_o)} + \frac{1}{k_1} W_1(t) + 0\{\phi(t)\}$$

where

$$\alpha = k_1[x_o - u_o - \tau(v_o - y_o)],$$

$$\beta = k_1 \tau (v_o - y_o),$$

$$\lambda = \sqrt{\alpha^2 + 4k_1 P_c},$$

$$W_o = \frac{1}{2}[k_1(x_o + u_o) - \lambda],$$

$$f(t) = \left[\frac{\beta}{2\tau}(\alpha\tau - 1) - k_1(P_c - v_o) + \frac{\beta^2}{4} e^{-\frac{1}{\tau}(t-t_o)}\right] e^{-\frac{1}{\tau}(t-t_o)},$$

$$W_1(t) = W_o\, e^{-\lambda(t-t_o)} + \int_{t_o}^{t} e^{-\lambda(t-s)} f(s)\, ds,$$

$$\phi(t) = \int_{t_o}^{t} e^{-\frac{\lambda}{2}(t-s)} f^2(s)\, ds,$$

and x_o, u_o, y_o, v_o are the values of the strengths of the offensive forces-in-being and the rates of production taken at time $t = t_o$ which, for any given positive $\epsilon \leq \frac{1}{2}$, are assumed to be such that $|W_1(t)| \leq \frac{\lambda\epsilon}{4}$ for $t \geq t_o$.

The symbol $0\{\phi(t)\}$ is used to denote a function $\psi(t)$ such that $|\psi(t)/\phi(t)| \leq K$ for $t \geq t_0$ where K is some positive constant.

COROLLARY 3: The strengths of the offensive forces-in-being approach finite limits as time increases indefinitely, namely

$$\lim_{t \to \infty} x(t) = \frac{1}{2k_1}(\lambda + \alpha),$$

$$\lim_{t \to \infty} u(t) = \frac{1}{2k_1}(\lambda - \alpha),$$

where

$$\alpha = k_1[x_0 - u_0 - \tau(v_0 - y_0)],$$

$$\lambda = \sqrt{\alpha^2 + 4k_1 P_c},$$

and x_0, u_0, v_0, y_0 are the values of the strengths of the offensive forces-in-being and the rates of production taken at some time $t = t_0 \geq 0$.

The preceding lemma yields the principal result of this section, which is more general than theorem 1. In that theorem it is stated that A defeats B provided at some time $x_0 > u_0$, $y_0 > v_0$. In view of the above lemma, however, this condition can be weakened considerably since it is clear that victory or defeat of A depends solely upon the sign of the constant α.

THEOREM 2: A necessary and sufficient condition for victory of A is that, at some time $t = t_0 \geq 0$, the strengths of the offensive forces-in-being and the rates of production satisfy the inequality

$$x_0 - u_0 > \tau(v_0 - y_0).$$

It follows as a consequence that the equation

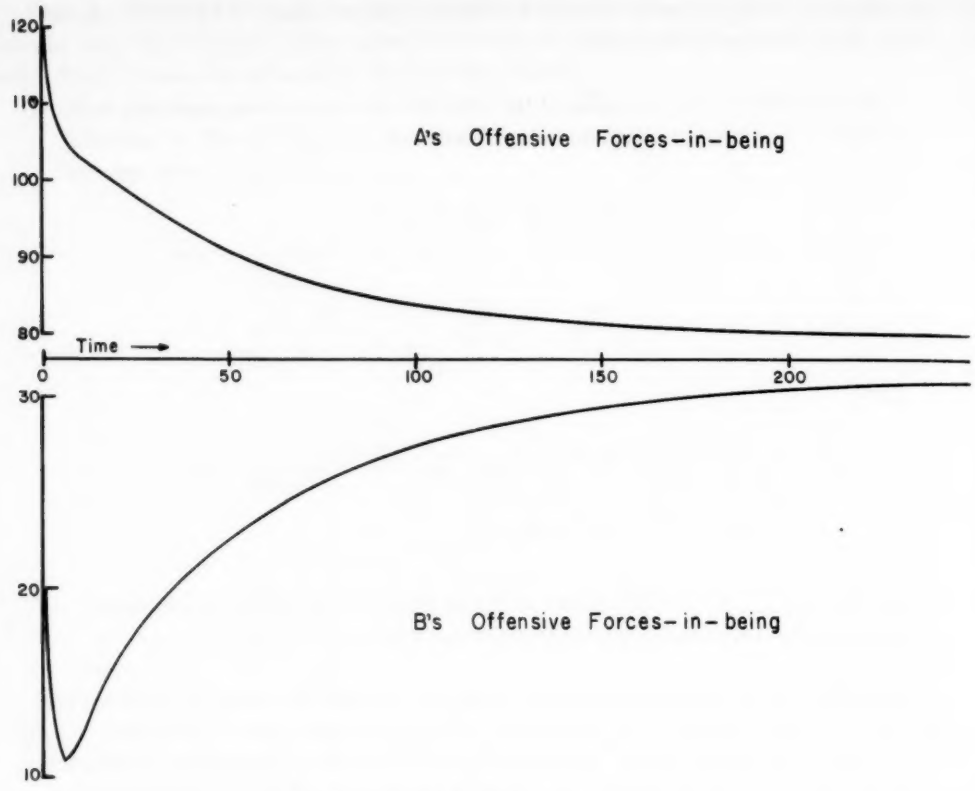
$$x_0 - u_0 = \tau(v_0 - y_0)$$

is both necessary and sufficient for an ultimate stalemate.

Figures 2 and 3 illustrate the above results. The values of the strengths of the offensive forces-in-being and the rates of production are at time $t = 0$ taken to be

$$x_0 = 120,\ y_0 = 2;\ u_0 = 20,\ v_0 = 3,$$

with the strategic parameters being $P_c = 5$, $\tau = 50$, $k_1 = 0.002$. (x, u have the dimension of military units, y, u, P_c the dimension of military unit per unit of time; τ has the dimension of



Figures 2 and 3

unit of time, $1/k_1$ the dimension of military unit times unit of time.) Thus, as time increases indefinitely,

$$x(t) \longrightarrow 80.9, \quad y(t) \longrightarrow 5,$$
$$u(t) \longrightarrow 30.9 \quad v(t) \longrightarrow 5,$$

so that A defeats B. Figure 2 shows that $x(t)$ is monotonically decreasing for all $t \geq 0$ whereas $u(t)$ is first decreasing to a minimum of about 10.986 units at time $t = 5.6$ (approximately) and thereafter increasing to its limit. This behavior is interesting from the point of view of numerical computations. For, in the absence of theoretical results, it might suggest that $u(t)$ would ultimately become larger than $x(t)$ so that B would defeat A, which is false. Of course, if numerical computations are carried out for sufficiently large times, the final outcome should become evident beyond doubt.

4. NECESSARY CONDITIONS FOR VICTORY

In this section we discuss the basic system (2.5) with a view towards establishing necessary conditions for victory of one party over its opponent. Victory shall be described again in terms of the magnitude of the strategic variables in accordance with the definitions (2.6) and (2.7).

For purposes of reference, we restate the system (2.5), namely,

$$\frac{dx}{dt} = y - bk_1 xu$$

$$\frac{dy}{dt} = \frac{1}{\tau}(P_c - y) - (1 - b) k_3 yu$$

$$\frac{du}{dt} = v - ak_1 ux$$

$$\frac{dv}{dt} = \frac{1}{\tau}(P_c - v) - (1 - a) k_3 vx$$

in which the variables and constants have the same meaning as before.

LEMMA 4: If the strength of B's offensive forces-in-being satisfies for all $t \geq t_o \geq 0$

(4.2) $$u(t) \geq u_o ,$$

then the strength of A's offensive forces-in-being and A's rate of production satisfy for all $t \geq t_o$

$$x(t) \leq \frac{Y}{\alpha} + \left(x_o - \frac{Y}{\alpha}\right) e^{-\alpha(t-t_o)}$$

$$y(t) \leq \frac{P_c}{\tau\beta} + \left(y_o - \frac{P_c}{\tau\beta}\right) e^{-\beta(t-t_o)}$$

where x_o is the strength of A's offensive forces-in-being, y_o A's rate of production at time $t = t_o$ and

$$\alpha = bk_1 u_o ,$$

$$\beta = \frac{1}{\tau} + (1 - b) k u_o ,$$

$$Y = \max\left\{y_o, \frac{P_c}{\tau\beta}\right\}.$$

LEMMA 5: If the strength of A's offensive forces-in-being satisfies for all $t \geq t_o \geq 0$

(4.5) $$x(t) \leq x_o$$

then the strength of B's offensive forces-in-being and B's rate of production satisfy for all $t \geq t_o$

$$u(t) \geq \frac{V}{\gamma} + \left(u_0 - \frac{V}{\gamma}\right) e^{-\gamma (t-t_0)}$$

$$v(t) \geq \frac{P_c}{\tau o} + \left(v_0 - \frac{P_c}{\tau \delta}\right) e^{-\delta (t-t_0)}$$

where u_0 is the strength of B's offensive forces-in-being, v_0 B's rate of production at time $t = t_0$, and

$$\gamma = a k_1 x_0 ,$$

$$\delta = \frac{1}{\tau} + (1 - a) k_3 x_0 ,$$

$$V = \min \left(v_0, \frac{P_c}{\delta}\right) .$$

Combining these two lemmas we obtain the following result.

THEOREM 3: In order that the strengths of A's and B's offensive forces-in-being and the rates of A's and B's production satisfy for all $t \geq t_0 \geq 0$ the inequalities

$$x(t) \leq x_0, \; y(t) \leq y_0 ,$$

$$u(t) \geq u_0, \; v(t) \geq v_0 , \tag{4.8}$$

where x_0, u_0 are the strengths of A's and B's offensive forces-in-being and y_0, v_0 the rates of A's and B's production at time $t = t_0$, it is necessary that

$$1 - \frac{P_c - v_0}{\tau k_3 x_0 v_0} \leq a \leq \frac{v_0}{k_1 x_0 u_0} ,$$

$$\frac{y_0}{k_1 u_0 x_0} \leq b \leq 1 - \frac{P_c - y_0}{\tau k_3 u_0 y_0} . \tag{4.9}$$

Several remarks are in order. In the first place, it is clear that these inequalities represent conditions upon the four strategic parameters k_1, k_3, a, b. The first two of these must be chosen such that

$$1 - \frac{P_c - v_0}{\tau k_3 x_0 v_0} \leq \frac{v_0}{k_1 x_0 u_0} , \qquad \frac{y_0}{k_1 u_0 x_0} \leq 1 - \frac{P_c - y_0}{\tau k_3 u_0 y_0} \tag{4.10}$$

and once these inequalities are satisfied, a, b can be selected so as to satisfy the inequalities (4.9). Secondly, from the derivation of these inequalities, which is given in Part II, p. 205, it will become apparent that the case $a = b = 1$ must be excluded here; it is covered completely by our previous discussion. Finally, the above theorem may be interpreted as yielding a necessary condition for victory of B over A. For if, at time $t = t_0 \geq 0$,

$$x_0 < u_0, \quad y_0 < v_0 \tag{4.11}$$

then in order that for all $t \geq t_0$

$$x(t) < u(t), \quad y(t) < v(t) \tag{4.12}$$

it is necessary that the inequalities (4.9) be satisfied. Note, however, that if (4.11) is satisfied, then, according to theorem 1, the strategy $a = b = 1$ (whatever k_1 and k_3) will guarantee victory of B over A.

In this connection let us point out another implication of the preceeding theorem. Suppose that x_0, u_0 are the strengths of A's and B's offensive forces-in-being and y_0, v_0 the rates of A's and B's production at some time $t = t_0 \geq 0$; and suppose, further, that the strategic parameters k_1 and k_3 satisfy the inequalities (4.10). Then selection of the parameters a, b outside the intervals I_1, I_2 respectively, where

$$I_1: \ 1 - \frac{P_c - v_0}{\tau k_3 x_0 v_0} \leq \xi \leq \frac{v_0}{k_1 x_0 u_0}, \qquad I_2: \ \frac{y_0}{k_1 u_0 x_0} \leq \xi \leq 1 - \frac{P_c - y_0}{\tau k_3 u_0 y_0}, \tag{4.13}$$

is sufficient to guarantee that the strengths of A's and B's offensive forces-in-being and the rates of A's and B's production do not satisfy for all $t \geq t_0$

$$x(t) \leq x_0, \quad y(t) \leq y_0, \qquad u(t) \geq u_0, \quad v(t) \geq v_0. \tag{4.14}$$

Thus, if in particular $x_0 < u_0$, $y_0 < v_0$, A will improve his chances to defeat B if the strategic parameters a, b do not fall within I_1, I_2, respectively. (Victory of A, however, is not assured thereby.)

5. ULTIMATE VALUES OF THE STRATEGIC VARIABLES

We now discuss some relations among the ultimate values of the strategic variables. Although these relations are derived under the assumption that the strength of one party's offensive forces-in-being tends to a limit as time increases indefinitely, we believe they are important enough to be included here.

Again, our discussion is based upon the system (2.5) (or (4.7)), namely the general model with offensive forces-in-being and production.

LEMMA 6: If the strength $u(t)$ of B's offensive forces-in-being tends to a limit u_∞ as time increases indefinitely, then the strength $x(t)$ of A's offensive forces-in-being, the rate $y(t)$ of A's production and the rate $v(t)$ of B's production tend, as time increases indefinitely, to the respective limits

$$x_\infty = \frac{P_c}{[1 + (1 - b)\, k_3\, \tau\, u_\infty]\, b k_1\, u_\infty},$$

(5.1)

$$y_\infty = \frac{P_c}{1 + (1 - b)\, k_3\, \tau\, u_\infty},$$

$$v_\infty = \frac{P_c}{1 + (1 - a)\, k_3\, \tau\, x_\infty}.$$

COROLLARY 6: If the strength of one party's offensive forces-in-being tends to a finite non-zero limit so does the strength of its opponent's offensive forces-in-being, the rates of its opponent's production as well as the rates of its own production.

In view of this lemma we can now formulate the following result.

THEOREM 4: If the strength $u(t)$ of B's offensive forces-in-being tends to a finite non-zero limit u_∞ as time increases indefinitely, and if the strategic parameters a, b are not equal, then the strength $x(t)$ of A's offensive forces-in-being, the rates $y(t)$ of A's production and the rates $v(t)$ of B's production tend, as time increases indefinitely, to the respective limits

$$x_\infty = \begin{cases} \dfrac{b - a}{(1 - a)\, a k_3\, \tau} + \dfrac{(1 - b)\, b}{(1 - a)\, a}\, u_\infty, & a \neq 1, \\ \dfrac{P_c}{k_1\, u_\infty}, & a = 1, \end{cases}$$

(5.2)

$$y_\infty = \frac{P_c}{1 + (1 - b)\, k_3\, \tau\, u_\infty},$$

$$v_\infty = \frac{a}{b}\, y_\infty.$$

This theorem may be used to obtain a sufficient condition for victory of one party over its opponent in terms of the strategic parameters only.

THEOREM 5: If the strength $u(t)$ of B's offensive forces-in-being tends to a finite non-zero limit u_∞ as time increases indefinitely, then A will defeat B provided the strategic parameters a, b satisfy

$$(0 \leq)\ a < b\ (\leq 1),$$

$$1 \leq a + b \leq 1 + \frac{1}{k_3 \tau u_\infty}. \tag{5.3}$$

Although the values of the strategic variables at some (finite) time $t = t_0 \geq 0$ do not enter explicitly into the inequalities (5.3) they are contained implicitly in u_∞.

PART II

This part is devoted to the mathematical derivation of the results stated in Part I. Throughout, our discussion is based upon the system of differential equations (2.5), namely

$$\begin{aligned}
\frac{dx}{dt} &= y - b k_1 x u \\
\frac{dy}{dt} &= \frac{1}{\tau}(P_c - y) - (1 - b) k_3 y u \\
\frac{du}{dt} &= v - a k_1 u x \\
\frac{dv}{dt} &= \frac{1}{\tau}(P_c - v) - (1 - a) k_3 v x
\end{aligned} \tag{0}$$

in which the parameters a, b, k_1, k_3, τ, P_c are assumed to be positive constants with a, b further restricted to be less than unity. From the general existence and uniqueness theorems it is clear that, given any t_0 and any x_0, y_0, u_0, v_0, this system possesses for $t \geq t_0$ and $t - t_0$ sufficiently small a unique solution $x(t)$, $y(t)$, $u(t)$, $v(t)$ satisfying $x(t_0) = x_0$, $y(t_0) = y_0$, $u(t_0) = u_0$, $v(t_0) = v_0$, which can be continued for all finite $t \geq t_0$.

1. NON-NEGATIVITY OF THE SOLUTIONS

From the second equation of (0) we obtain, since $P_c/\tau > 0$,

$$\frac{dy}{dt} \geq -\left[\frac{1}{\tau} + (1 - b) k_3 u\right] y \tag{1.1}$$

or

$$\frac{d}{dt}\left\{y \exp\left[\frac{1}{\tau}(t - t_0) + (1 - b) k_3 \int_{t_0}^{t} u(s)\, ds\right]\right\} \geq 0 \tag{1.2}$$

whence

$$y(t) \geq y_0 \exp\left\{-\frac{1}{\tau}(t - t_0) - (1 - b) k_3 \int_{t_0}^{t} u(s)\, ds\right\}. \tag{1.3}$$

Thus, if $y_0 \geq 0$ then $y(t) \geq 0$. Similarly,

$$v(t) \geq v_0 \exp\left\{-\frac{1}{\tau}(t - t_0) - (1 - a) k_3 \int_{t_0}^{t} x(s)\, ds\right\} \tag{1.4}$$

whence, if $v_0 \geq 0$, then $v(t) \geq 0$.

It follows from the first equation of (0) that if $y(t) \geq 0$, then

$$\frac{dx}{dt} \geq -b k_1 x u \tag{1.5}$$

or

$$\frac{d}{dt}\left\{x \exp b k_1 \int_{t_0}^{t} u(s)\, ds\right\} \geq 0 \tag{1.6}$$

whence

$$x(t) \geq x_0 \exp\left\{b k_1 \int_{t_0}^{t} u(s)\, ds\right\}. \tag{1.7}$$

Thus, if $x_0 \geq 0$ then $x(t) \geq 0$. Similarly, if $v(t) \geq 0$ then

$$u(t) \geq u_0 \exp\left\{-a k_1 \int_{t_0}^{t} x(s)\, ds\right\} \tag{1.8}$$

whence, if $u_0 \geq 0$, then $u(t) \geq 0$.

If $y(t) \geq 0$, $u(t) \geq 0$ for all $t \geq t_0$ then we find from the second equation of (0) that

$$\frac{dy}{dt} \leq \frac{1}{\tau}(P_c - y) \tag{1.9}$$

whence

$$y(t) \leq P_c - (P_c - y_0)\, e^{-\frac{1}{\tau}(t - t_0)} \tag{1.10}$$

so that, in particular, $y(t) \le P_c$ for $t \ge t_o$ provided $y_o \le P_c$. Similarly, if $v(t) \ge 0$, $x(t) \ge 0$ for all $t \ge t_o$, then

$$v(t) \le P_c - (P_c - v_o)\, e^{-\frac{1}{\tau}(t-t_o)} \tag{1.11}$$

and so $v(t) \le P_c$ for $t \ge t_o$ provided $v_o \le P_c$.

Summarizing, we can state, therefore, if at $t = t_o$

$$\begin{gathered} x(t_o) \ge 0, \quad u(t_o) \ge 0 \\ 0 \le y(t_o) \le P_c, \quad 0 \le v(t_o) \le P_c \end{gathered} \tag{1.12}$$

then for all $t \ge t_o$

$$\begin{gathered} x(t) \ge 0, \quad u(t) \ge 0 \\ 0 \le y(t) \le P_c, \quad 0 \le v(t) \le P_c . \end{gathered} \tag{1.13}$$

2. THE CASE a = b = 1

In this case the system (0) reduces to the four equations

$$\begin{aligned} \frac{dx}{dt} &= y - k_1\, x\, u \\ \frac{dy}{dt} &= \frac{1}{\tau}(P_c - y) \\ \frac{du}{dt} &= v - k_1\, u\, x \\ \frac{dv}{dt} &= \frac{1}{\tau}(P_c - v) \end{aligned} \tag{2.1}$$

of which the second and fourth can be solved immediately.

We find

$$\begin{aligned} y(t) &= P_c - (P_c - y_o)\, e^{-\frac{1}{\tau}(t-t_o)} \\ v(t) &= P_c - (P_c - v_o)\, e^{-\frac{1}{\tau}(t-t_o)} \end{aligned} \tag{2.2}$$

which is lemma 1.

Subtracting the third equation from the first in (0) and integrating, we obtain

$$x(t) - u(t) = x_0 - u_0 + \int_{t_0}^{t} (y(s) - v(s))\, ds \tag{2.3}$$

whence by (2.2)

$$x(t) - u(t) = x_0 - u_0 + \tau(y_0 - v_0)\,(1 - e^{-\frac{1}{\tau}(t-t_0)}). \tag{2.4}$$

Introducing this expression into the third equation of (0) we can eliminate $x(t)$ to obtain the general Riccati equation

$$\frac{du}{dt} = -k_1 u^2 - k_1 u[x_0 - u_0 + \tau(y_0 - v_0)\,(1 - e^{-\frac{1}{\tau}(t-t_0)})] + v(t) \tag{2.5}$$

where $v(t)$ is given by (2.2). To simplify our notation, let us put

$$\begin{aligned}
\alpha &= k_1 [x_0 - u_0 + \tau (y_0 - v_0)]\,, \\
\beta &= k_1 \tau (v_0 - y_0)\,, \\
\gamma &= k_1 (P_c - v_0)\,, \\
\lambda &= \sqrt{\alpha^2 + 4k_1 P_c}\,, \\
\mu &= \frac{1}{4}\beta^2\,, \\
\nu &= \frac{\beta}{2\tau}(\alpha\tau - 1) - \gamma\,,
\end{aligned} \tag{2.6}$$

and introduce the new variable

$$w(t) = k_1 u(t) + \frac{1}{2}(\alpha - \lambda) + \frac{1}{2}\beta\, e^{-\frac{1}{\tau}(t-t_0)}. \tag{2.7}$$

Then, after considerable simplification, we find

$$\frac{dw}{dt} = -w^2 - \lambda w + (\mu e^{-\frac{1}{\tau}(t-t_0)} + \nu)\, e^{-\frac{1}{\tau}(t-t_0)} \tag{2.8}$$

i.e.,

$$\frac{dw}{dt} = -w^2 - \lambda w + f(t) \tag{2.9}$$

where

$$f(t) = (\mu\, e^{-\frac{1}{\tau}(t-t_o)} + \nu)\, e^{-\frac{1}{\tau}(t-t_o)}. \tag{2.10}$$

Note that $f(t) \to 0$ as $t \to \infty$ and $\int_{t_o}^{\infty} |f(t)|^n \, dt < \infty$ for any $n > 0$.

We shall solve the equation (2.9) by the method of successive approximations under the assumption that at $t = t_o$

$$w(t_o) \equiv w_o = k_1 u_o + \frac{1}{2}(\alpha + \beta - \lambda). \tag{2.11}$$

Let us write

$$w_1(t) = w_o e^{-\lambda(t-t_o)} + \int_{t_o}^{t} e^{-\lambda(t-s)} f(s)\, ds,$$

$$w_{n+1}(t) = w_1(t) - \int_{t_o}^{t} e^{-\lambda(t-s)} w_n^2(s)\, ds, \quad n = 1, 2, \ldots \tag{2.12}$$

To prove convergence of this sequence, we first show that for all $t \geq t_0$ and all $n \geq 0$

$$|w_n(t)| \leq \frac{\lambda\epsilon}{2}, \quad 0 < \epsilon \leq 1 \tag{2.13}$$

provided that for $t \geq t_o$

$$|w_1(t)| \leq \frac{\lambda\epsilon}{4}. \tag{2.14}$$

There is no loss of generality in assuming (2.14) to hold for $t \geq t_o$. We have, by induction

$$|w_{n+1}(t)| \leq |w_1(t)| + \int_{t_o}^{t} e^{-\lambda(t-s)} |w_n(s)|^2 \, ds$$

$$\leq \frac{\lambda\epsilon}{4} + \frac{\lambda^2\epsilon^2}{4} \int_{t_o}^{t} e^{-\lambda(t-s)} \, ds \leq \frac{\lambda\epsilon}{2} \tag{2.15}$$

provided $0 < \epsilon \leq 1$. Now

$$|w_{n+1}(t)| \leq |w_1(t)| + \frac{\lambda\epsilon}{2} \int_{t_o}^{t} e^{-\lambda(t-s)} |w_n(s)| \, ds. \tag{2.16}$$

We shall show next that

$$|w_{n+1}(t)| \le 2\,|w_o|\, e^{-\frac{\lambda}{2}(t-t_o)} + 2\int_{t_o}^{t} e^{-\frac{\lambda}{2}(t-s)}\,|f(s)|\,ds\,. \tag{2.17}$$

where $f(t)$ is defined as in (2.10). Certainly, we have from (2.12)

$$\begin{aligned} |w_1(t)| &\le |w_o|\, e^{-\lambda(t-t_o)} + \int_{t_o}^{t} e^{-\lambda(t-s)}\,|f(s)|\,ds \\ &\le 2\,|w_o|\, e^{-\frac{\lambda}{2}(t-t_o)} + 2\int_{t_o}^{t} e^{-\frac{\lambda}{2}(t-s)}\,|f(s)|\,ds\,. \end{aligned} \tag{2.18}$$

By induction, we find, using (2.16)

$$\begin{aligned} |w_{n+1}(t)| &\le |w_1(t)| + \frac{\lambda\epsilon}{2}\int_{t_o}^{t} 2\,|w_o|\, e^{-\frac{\lambda}{2}(s-t_o)}\, e^{-\lambda(t-s)}\,ds \\ &\quad + \frac{\lambda\epsilon}{2}\int_{t_o}^{t} e^{-\lambda(t-s)}\int_{t_o}^{s} e^{-\frac{\lambda}{2}(s-v)}\,|f(v)|\,dv\,ds \\ &\le |w_o|\, e^{-\frac{\lambda}{2}(t-t_o)} + \int_{t_o}^{t} e^{-\frac{\lambda}{2}(t-s)}\,|f(s)|\,ds \\ &\quad + \lambda\epsilon\,|w_o|\int_{t_o}^{t} e^{-\frac{\lambda}{2}(s-t_o)}\, e^{-\lambda(t-s)}\,ds \\ &\quad + \lambda\epsilon\int_{t_o}^{t} e^{-\lambda(t-s)}\int_{t_o}^{s} e^{-\frac{\lambda}{2}(s-v)}\,|f(v)|\,dv\,ds\,. \end{aligned} \tag{2.19}$$

It is not difficult to show that the last two integrals on the right are majorized by

$$2\epsilon\left\{|w_o|\, e^{-\frac{\lambda}{2}(t-t_o)} + \int_{t_o}^{t} e^{-\frac{\lambda}{2}(t-s)}\,|f(s)|\,ds\right\} \tag{2.20}$$

whence (2.19) can be rewritten in the form

$$|w_{n+1}(t)| \le (1+2\epsilon)\left\{|w_o|\ e^{-\frac{\lambda}{2}(t-t_o)} \int_{t_o}^{t} e^{-\frac{\lambda}{2}(t-s)}\ |f(s)|\ ds\right\}$$

(2.21)

$$\le 2\ |w_o|\ e^{-\frac{\lambda}{2}(t-t_o)} + 2\int_{t_o}^{t} e^{-\frac{\lambda}{2}(t-s)}\ |f(s)|\ ds$$

provided $0 < \epsilon \le \frac{1}{2}$. Thus (2.17) holds for all integers $n \ge 1$.

To conclude the proof of convergence of the sequence $w_n(t)$, note that it converges if the series $\Sigma\,[w_{n+1}(t) - w_n(t)]$ converges. Hence it is sufficient to show the convergence of the series $\Sigma\,|w_{n+1}(t) - w_n(t)|$. By use of (2.13) it is easy to show that

(2.22) $$|w_{n+1}(t) - w_n(t)| \le \frac{\lambda \epsilon^{n+1}}{4}$$

so that

(2.23) $$\Sigma\,|w_{n+1}(t) - w_n(t)| << \Sigma\,\frac{\lambda}{4}\epsilon^{n+1}$$

where, by assumption, $0 < \epsilon \le \frac{1}{2}$. Thus, the series on the left of (2.23) converges uniformly for all $t \ge t_o$. It is clear that

(2.24) $$w(t) = \lim_{n\to\infty} w_n(t)$$

exists and satisfies the equation (2.9). Moreover,

(2.25) $$|w(t)| \le 2\ |w_o|\ e^{-\frac{\lambda}{2}(t-t_o)} + 2\int_{t_o}^{t} e^{-\frac{\lambda}{2}(t-s)}\ |f(s)|\ ds\,.$$

Hence, by Schwarz's inequality, we have

$$w^2(t) \le 2\left\{4w_o^{\,2}\ e^{-\lambda(t-t_o)} + 4\left[\int_{t_o}^{t} e^{-\frac{\lambda}{2}(t-s)}\ |f(s)|\ ds\right]^2\right\}$$

(2.26)

$$\le 8w_o^{\,2}\ e^{-\lambda(t-t_o)} + \frac{16}{\lambda}\int_{t_o}^{t} e^{-\frac{\lambda}{2}(t-s)}\ f^2(s)\ ds\,.$$

It follows that

$$\int_{t_0}^{t} e^{-\lambda(t-s)}\, w^2(s)\, ds \le 8 w_0^2 \int_{t_0}^{t} e^{-\lambda(t-s)}\, e^{-\lambda(s-t_0)}\, ds$$

$$+ \frac{16}{\lambda} \int_{t_0}^{t} e^{-\lambda(t-s)} \int_{t_0}^{s} e^{-\frac{\lambda}{2}(s-v)}\, f^2(v)\, dv\, ds \tag{2.27}$$

$$\le 8 w_0^2 (t-t_0)\, e^{-\lambda(t-t_0)} + \frac{32}{\lambda^2} \int_{t_0}^{t} e^{-\frac{\lambda}{2}(t-s)}\, f^2(s)\, ds .$$

Hence, we have

$$\int_{t_0}^{t} e^{-\lambda(t-s)}\, w^2(s)\, ds = 8 w_0^2 (t-t_0)\, e^{-\lambda(t-t_0)} + 0\left\{\int_{t_0}^{t} e^{-\frac{\lambda}{2}(t-s)}\, f^2(s)\, ds\right\}. \tag{2.28}$$

Thus we find using (2.12)

$$w(t) = w_0\, e^{-\lambda(t-t_0)} + \int_{t_0}^{t} e^{-\lambda(t-s)}\, f(s)\, ds$$

$$- 8 w_0^2 (t-t_0)\, e^{-\lambda(t-t_0)} + 0\left\{\int_{t_0}^{t} e^{-\frac{\lambda}{2}(t-s)}\, f^2(s)\, ds\right\}, \tag{2.29}$$

and this expression gives a complete description of the behavior of $w(t)$ as $t \to \infty$.

From (2.7) we obtain

$$u(t) = \frac{1}{2k_1}(\lambda - \alpha) - \frac{1}{2k_1}\beta\, e^{-\frac{1}{\tau}(t-t_0)} + \frac{1}{k_1}\, w(t), \tag{2.30}$$

whence, using (2.29),

$$u(t) = \frac{1}{2k_1}(\lambda - \alpha) - \frac{8 w_0^2}{k_1}(t - t_0)\, e^{-\lambda(t-t_0)} - \frac{\beta}{2k_1}\, e^{-\frac{1}{\tau}(t-t_0)}$$

$$+ \frac{1}{k_1}\, w_0\, e^{-\lambda(t-t_0)} + \frac{1}{k_1} \int_{t_0}^{t} e^{-\lambda(t-s)}\, f(s)\, ds + 0\{\phi(t)\} \tag{2.31}$$

where

$$\phi(t) = \int_{t_0}^{t} e^{-\frac{\lambda}{2}(t-s)} f^2(s)\, ds . \tag{2.32}$$

Finally, from (2.4) it follows that

$$x(t) = \frac{1}{2k_1}(\lambda + \alpha) - \frac{8w_0^2}{k_1}(t - t_0)\, e^{-\lambda(t-t_0)} + \frac{\beta}{2k_1} e^{-\frac{1}{\tau}(t-t_0)}$$

$$+ \frac{1}{k_1} w_0\, e^{-\lambda(t-t_0)} + \frac{1}{k_1}\int_{t_0}^{t} e^{-\lambda(t-s)} f(s)\, ds + 0\{\phi(t)\} .$$

Equations (2.31) and (2.33) make up lemma 3.

3. SOME INEQUALITIES ON THE DEPENDENT VARIABLES

We consider now the system (0) and assume that $u(t) \geq u_0 > 0$ for $t \geq t_0 \geq 0$. Then, if $y_0 \geq 0$ (so that $y(t) \geq 0$ for $t \geq t_0$), we find

$$\frac{dy}{dt} \leq \frac{1}{\tau}(P_c - y) - (1 - b) k_3 u_0 y \tag{3.1}$$

whence by integration

$$y(t) \leq \frac{P_c}{\tau\beta} + \left(y_0 - \frac{P_c}{\tau\beta}\right) e^{-\beta(t-t_0)} \tag{3.2}$$

where

$$\beta = \frac{1}{\tau} + (1 - b) k_3 u_0 . \tag{3.3}$$

Let $Y = \max\{y_0, P_c/\tau\beta\}$; then for all $t \geq t_0$

$$0 \leq y(t) \leq Y \tag{3.4}$$

provided $y_0 \geq 0$.

In view of (3.4) we find from (0)

$$\frac{dx}{dt} \leq Y - bk_1 u_0 x \tag{3.5}$$

provided $x_0 \geq 0$ (so that $x(t) \geq 0$ for $t \geq t_0$); hence by integration

$$x(t) \leq \frac{Y}{\alpha} + \left(x_0 - \frac{Y}{\alpha}\right) e^{-\alpha(t-t_0)} \tag{3.6}$$

where

$$\alpha = b k_1 u_0 . \tag{3.7}$$

Let $X = \max\{x_0, Y/\alpha\}$; then for all $t \geq t_0$

$$0 \leq x(t) \leq X \tag{3.8}$$

provided $x_0 \geq 0$.

In view of (3.8) we now find from (0)

$$\frac{dv}{dt} \geq \frac{1}{\tau}(P_c - v) - (1 - a) k_3 X v \tag{3.9}$$

provided $v_0 \geq 0$ (so that $v(t) \geq 0$ for $t \geq t_0$); hence by integration

$$v(t) \geq \frac{P_c}{\tau\delta} + \left(v_0 - \frac{P_c}{\tau\delta}\right) e^{-\delta(t-t_0)} \tag{3.10}$$

where

$$\delta = \frac{1}{\tau} + (1 - a) k_3 X . \tag{3.11}$$

Putting $V = \min\{v_0, P_c/\tau\delta\}$ we have for all $t \geq t_0$

$$V \leq v(t) \leq P_c . \tag{3.12}$$

Finally, we obtain from (0) by using (3.12)

$$\frac{du}{dt} \geq V - a k_1 X u \tag{3.13}$$

since $u(t) \geq 0$ for $t \geq t_0$ by virtue of the assumption $u_0 > 0$. Hence

$$u(t) \geq \frac{V}{\gamma} + \left(u_0 - \frac{V}{\gamma}\right) e^{-\gamma(t-t_0)} \tag{3.14}$$

where

$$\gamma = a k_1 X . \tag{3.15}$$

Let $U = \min(u_o, V/\gamma)$; then for all $t \geq t_o$

$$u(t) \geq U. \tag{3.16}$$

Since our basic assumption was that $u(t) \geq u_o$ for all $t \geq t_o$ we must have

$$u_o \leq \frac{V}{\gamma} \tag{3.17}$$

so that $U = u_o$, for otherwise (3.16) would be impossible.

Combining the inequalities (3.4), (3.8), (3.12), (3.17) we can conclude that if $u(t) \geq u_o > 0$ for $t \geq t_o$ then

$$\begin{aligned} 0 \leq x(t) &\leq X_o \\ 0 \leq y(t) &\leq Y_o \\ v_o \leq v(t) &\leq P_c \end{aligned} \tag{3.18}$$

provided $x_o > 0$, $y_o > 0$, $v_o > 0$ and

$$\begin{aligned} \frac{P_c}{1 + (1 - b)\tau k_3 u_o} \leq y_o,\ x_o \geq \frac{y_o}{bk_1 u_o}, \\ \frac{P_c}{1 + (1 - a)\tau k_3 x_o} \geq v_o,\ u_o \leq \frac{v_o}{ak_1 x_o}. \end{aligned} \tag{3.19}$$

Considering x_o, y_o, u_o, v_o as fixed (given) constants, this system represents a system of inequalities on the parameters a, b, k_1, k_3, τ, P_c, namely

$$\begin{aligned} \frac{P_c - y_o}{\tau k_3 u_o y_o} \leq 1 - b,\ b \geq \frac{y_o}{k_1 x_o u_o}, \\ \frac{P_c - v_o}{\tau k_3 x_o v_o} \geq 1 - a,\ a \leq \frac{v_o}{k_1 x_o u_o}, \end{aligned} \tag{3.20}$$

or

$$\begin{aligned} 1 - \frac{P_c - v_o}{\tau k_3 x_o v_o} \leq a \leq \frac{v_o}{k_1 x_o u_o}, \\ \frac{y_o}{k_1 x_o u_o} \leq b \leq 1 - \frac{P_c - y_o}{\tau k_3 u_o y_o}. \end{aligned} \tag{3.21}$$

Thus, if $u(t) \geq u_0 > 0$ for $t \geq t_0$ and if $x_0 > 0$, $y_0 > 0$, $v_0 > 0$, then in order that the inequalities (3.18) hold for $t \geq t_0$ it is necessary that the parameters a, b, k_1, k_3, τ, P_c satisfy the inequalities (3.21). This is theorem 3.

Suppose now that as $t \to \infty$

$$u(t) \to u_\infty > 0 . \tag{3.22}$$

From the second equation of (0) we find, by integrating it as a linear first order equation in $y(t)$, that

$$y(t) = y_0 e^{-\phi_1(t)} + \frac{1}{\tau} P_c e^{-\phi_1(t)} \int_{t_0}^{t} e^{\phi_1(s)} \, ds \tag{3.23}$$

where

$$\phi_1(t) = \frac{1}{\tau}(t - t_0) + (1 - b) k_3 \int_{t_0}^{t} u(s) \, ds . \tag{3.24}$$

Similarly, from the first equation of (0) we obtain

$$x(t) = x_0 e^{-\psi_1(t)} + e^{-\psi_1(t)} \int_{t_0}^{t} y(s) e^{\psi_1(s)} \, ds \tag{3.25}$$

where

$$\psi_1(t) = b k_1 \int_{t_0}^{t} u(s) \, ds . \tag{3.26}$$

Therefore,

$$y_\infty \equiv \lim_{t \to \infty} y(t) = \frac{1}{\tau} P_c \lim_{t \to \infty} e^{-\phi_1(t)} \int_{t_0}^{t} e^{\phi_1(s)} \, ds = \frac{P_c}{1 + (1 - b) \tau k_3 u_\infty} \tag{3.27}$$

and, since $\int_{t_0}^{\infty} u(s) \, ds = \infty$,

$$x_\infty \equiv \lim_{t \to \infty} x(t) = \lim_{t \to \infty} e^{-\psi_1(t)} \int_{t_0}^{t} y(s) e^{\psi_1(s)} \, ds = \frac{P_c}{[1 + (1 - b) \tau k_3 u_\infty] b k_1 u_\infty} . \tag{3.28}$$

Analogously, we find from (0)

$$u(t) = u_o\, e^{-\psi_2(t)} + e^{-\psi_2(t)} \int_{t_o}^{t} v(s)\, e^{\psi_2(s)}\, ds\,,$$

(3.29)

$$v(t) = v_o\, e^{-\phi_2(t)} + \frac{1}{\tau} P_c\, e^{-\phi_2(t)} \int_{t_o}^{t} e^{\phi_2(s)}\, ds$$

where

$$\phi_2(t) = \frac{1}{\tau}(t - t_o) + (1 - a) k_3 \int_{t_o}^{t} x(s)\, ds\,,$$

(3.30)

$$\psi_2(t) = a k_1 \int_{t_o}^{t} x(s)\, ds\,.$$

Thus, if $x_\infty > 0$ then

$$u_\infty \equiv \lim_{t\to\infty} u(t) = \lim_{t\to\infty} e^{-\psi_2(t)} \int_{t_o}^{t} v(s)\, e^{\psi_2(s)}\, ds = \frac{P_c}{[1 + (1 - a)\tau k_3 x_\infty] a k_1 x_\infty}\,,$$

(3.31)

$$v_\infty \equiv \lim_{t\to\infty} v(t) = \frac{1}{\tau} P_c \lim_{t\to\infty} e^{-\phi_2(t)} \int_{t_o}^{t} e^{\phi_2(s)}\, ds = \frac{P_c}{1 + (1 - a)\tau k_3 x_\infty}\,.$$

In view of (3.27), (3.28), (3.31) we can conclude that if $u(t) \to u_\infty > 0$ as $t \to \infty$ then $x(t) \to x_\infty$ as $t \to \infty$ where

(3.32)
$$x_\infty u_\infty = \frac{1}{k_1 a} \frac{P_c}{1 + (1 - a)\tau k_3 x_\infty} = \frac{1}{k_1 b} \frac{P_c}{1 + (1 - b)\tau k_3 u_\infty}$$

Hence, if $a < 1$ and $b \neq a$, then

$$x_\infty = \frac{b - a}{(1 - a)\, a\, k_3\, \tau} + \frac{(1 - b)\, b}{(1 - a)\, a}\, u_\infty$$

(3.33)
$$y_\infty = \frac{P_c}{1 + (1 - b)\, k_3\, \tau\, u_\infty}$$

$$v_\infty = \frac{a}{b}\, \frac{P_c}{1 + (1 - b)\, k_3\, \tau\, u_\infty} = \frac{a}{b}\, y_\infty$$

This is theorem 5.

REMARKS

There appear to be essentially two directions in which the present investigation could be continued. One that suggests itself naturally from the results we have obtained here would focus attention on a more suitable definition of victory not based upon the ultimate values of the strategic variables as time increases indefinitely. Indeed, as has been pointed out in the Introduction, the ultimacy statement in our definition of victory is a definite drawback, as a consequence of which our results are not as realistic as might be hoped (and this will remain true however well the model approximates actual situations). Perhaps a more natural way to describe victory in terms of the magnitude of the strategic variables might be to define simply as victor that party whose strategic variables exceed at some time those of its opponents. In this case a mathematical investigation could not draw upon the many available results on the asymptotic behavior of solutions of systems of nonlinear differential equations and thus would likely be considerably more difficult. On the other hand it would appear that any numerical investigation on a high-speed computer such as SEAC, the National Bureau of Standards' Eastern Automatic Computer, would not be more complicated.

Another direction further research could take would be toward modification of the present model so as to achieve more realism. In this connection it would seem natural to abandon the restrictive assumption made here that both parties have the same goal for their rates of production and the same time constant. Although this modification may appear minor, it would certainly affect our results considerably. Since some of our present techniques would still be applicable, such a change would not seem to introduce more difficulties. Another modification might be based upon a different concept of attrition which would introduce in the terms due to attrition more complicated functions of the strategic variables. This is certain to increase mathematical difficulties greatly, so that perhaps numerical investigations of an exploratory nature should be carried out before any theoretical study is undertaken.

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NEWS AND MEMORANDA

Readers are invited to submit to the Managing Editor items of general interest in the field of logistics.

A seminar on Mathematics in Management was held at Pennsylvania State University during the week of June 13, 1955, under the chairmanship of Professor Norman David. The speakers included representatives from universities, industrial firms and management consultants. The topics discussed ranged from the personnel and organizational requirements for an operations research staff to technical questions of production and inventory control.

The First Annual Statistical Engineering Symposium sponsored by the Chemical Corps Engineering Agency was held at the Army Chemical Center, Maryland, on April 28 and 29, 1955. Most of the speeches dealt with the design and analysis of experiments.

Colonel T. B. Evans, U. S. Army, has become an Associate Editor of the Naval Research Logistics Quarterly.

The Management Sciences Research Project, sponsored at UCLA by the Logistics Branch, Office of Naval Research, is engaged in research on industrial production planning and scheduling at the single-firm level. Its work is directed toward the development of systematic techniques for solving problems in this area, and emphasizes the use of mathematical methods and the application of electronic data-handling equipment. The project is a small interdisciplinary team, under the directorship of Professor A. J. Rowe of the School of Business Administration. It is jointly sponsored by the Department of Mathematics, the College of Engineering, and the School of Business Administration.

Good production planning is always valuable to industry, and becomes especially important during mobilization, when changeover and increased production rates multiply the difficulties. Project research has largely been guided by these considerations; however, the results of this work are widely applicable to problems arising in peacetime production.

The area of research being undertaken by the project can be divided into two interrelated parts, long-range planning and prediction, and detail scheduling necessary to effectuate overall plans. These two phases should ultimately be considered together, but most of the research to date has treated them as separate problems.

The scope of production planning problems has led the project into research on related problems such as communications, electronic data handling, and work measurement. The specific topics being considered are dealt with in the following paragraphs.

1. Master planning. The over-all production planning problems of a firm have been studied mainly by the use of linear programming. This research began with mathematical models representing the capacity of a plant or a firm in terms of processes; thus making it possible to evaluate changeover possibilities, either from the point of view of potential production of new products or from that of the additional facilities needed to accomplish enlarged production goals. Other, more dynamic production models have also been developed, but these have generally been too large for direct application. In order to develop the methodology for the process type model, specific processes encountered in the basic production of iron and steel have been studied. A model of a steel mill which has been developed is adapted to estimating the mill's capacity for producing various product mixes under given supply conditions, to determining the most economical way to produce a bill of goods under given supply and price conditions, and various other problems.

Current work on the iron and steel production model is centered on further refinement which will more accurately represent specific plants, and which will provide an integrated model including ingot production, rolling mill production, and finishing processes. This model is small enough to permit practical computations, and a test application which will indicate its usefulness at the plant level has begun with a major steel producer.

2. Detail scheduling. Project research in this area has centered around the development and testing of "scheduling rules" for systematically assigning the sequence of jobs on a given machine. The selection of routings, lot-size determination, and transportation problems within a plant have also been studied in relation to the central question. Research on scheduling rules has taken two principal directions, the first toward the mathematical analysis and solution of comparatively simple problems, and the second toward the development of methods applicable to extremely complex production situations such as are typical of job-shop production.

A major difficulty encountered in the mathematical scheduling problems stems from their essentially combinatorial character, brought about by the incremental costs and machine-time consumption resulting from "lot splitting." Several types of problems have been solved or partially solved, in which a number of job-lots must be sequenced through a single machine or production-line, subject to various restrictions, so as to achieve some specific goal; and some very special problems involving two or more machines have been successfully handled mathematically.

The study of complex, large-scale problems has been approached through the examination of methods which appear effective in actual practice, and through the study of mathematically tractable sub-problems. Various classes of decision rules have been developed, and these have been compared and evaluated by simulated application. Current research is concerned with further testing of decision rules, particularly with regard to their effectiveness when using rough estimates of such parameters as the time for machining. The methods developed are computationally simple, and a practical feasibility test will be carried out at a medium-size job-shop in the near future.

3. Data processing. The difficulties arising in programming computations for planning and scheduling problems have led to the design of special purpose equipment for these and related problems. Other research has been concerned with the problem of designing suitable communication systems for the application of various production planning and scheduling methods.

Present research is in the area of systems analysis, with the emphasis on development of methods for improving the representation of information flow and on a better understanding of the functions and characteristics of information feedback in manufacturing operations.

4. Work measurement. Because of the need for predictions of machine processing times, times required for transportation, etc., the project has undertaken some studies in work measurement, particularly in ratio delay theory and in synthetic time study. A study now under way is concerned with testing the assumption of independence of elemental times in the synthesis of standard times for sequences of motions.

Project members have presented numerous papers to scholarly and technical societies, and many of these have been published in scientific and technical journals. The work on data-processing will be published in book form in the near future. Project research reports are widely distributed by request to industrial firms and to universities, and a bibliography of these reports is available.

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RECENT PUBLICATIONS

JAPAN'S DECISION TO SURRENDER. By Robert J. C. Butow, Stanford University Press, 1954, 259 pp.

Japan's acceptance of the Potsdam ultimatum on August 14, 1945 represented much more than just a hasty reaction to the atomic bomb and to the Soviet Union's entry into the war. The capitulation was actually the climax of the behind-the-scenes activities of those Japanese whose thoughts and actions, over an extended period, produced Japan's decision to surrender. This book is a documented history of those activities and a basic study of that decision — the first to appear in the English language.

The narrative begins with the early months of the military's gamble at conquest, keeps pace with the successive crises growing out of Japan's mounting defeats, and terminates with the unprecedented developments which keynote the last fateful days of the war. The internal obstacles encountered by the end-the-war advocates and the problems posed from the outside by Allied policy pronouncements figure both as incidents in Japan's struggle to surrender and as examples of why the surrender was a struggle.

The book is based on hitherto untapped primary sources including the records of the International Military Tribunal for the Far East, documents in the possession of and investigations undertaken by the Japanese Ministry of Foreign Affairs, unclassified and unpublished interrogation files of the United States Army, and personal material (notes, diaries, letters) obtained by the author from a number of the participants in the events described.

FALSCH UND RICHTIG PLANEN. EINE KRITISCHE STUDIE UBER DIE DEUTSCHE WIRTSCHAFTSLENKUNG IN ZWEITEN WELTKRIEG. By Erich Welter, Quelle und Meyer, Heidelberg, 1954, 164 pp.
(False and Correct Planning. A critical examination of German economic planning during the second World War).

This short book is the only analysis, available so far, of the German experiences with economic planning during the last war. The author is a well known economist. He discusses the failures of the various German plans, such as there were. He shows how rationing and priority rating worked, how one stop-gap plan or scheme followed another, how proper statistical information was lacking and how even the need for numerical information was recognized only very late, how the bombings disrupted the economy, how contradictions of planning developed at various levels of decision making. All this is shown by means of many examples some of which are subtle, others, because of our hindsight, outright hair-raising. Many examples could also apply to other countries, so great is the similarity in some experiences. For example, the constant raising of priorities was as frequent in Germany as in the United States. The book shows that a convincing theory of planning is still lacking and that to discover it will be a very difficult task.

STRATEGIES ET DECISIONS ECONOMIQUES. Etudes théoriques et applications aux entreprises. By G. Th. Guilbaud, P. Massé, and R. Hénon, Editions du Centre National de la Recherche Scientifique, Paris, 1954, 245 pp.

This book comprises a series of lectures presented at the Econometric Seminar of the National Center of Scientific Research, Paris, on the problems of decision-making and the industrial applications of the theory of games. G. Th. Guilbaud writes mostly about the rational and the successive developments of the theory of games. This presentation, which takes up the first 125 pages of the book, is quite original and enables its readers to grasp quickly the essentials of the theory of games. The next 99 pages by P. Massé are centered around the problem of the optimum administration of the French economy and more particularly of the French Power Supply System of which P. Massé is the General Director. His interest in engineering as well as in economics will be appreciated by the readers interested in logistics. Finally, R. Hénon, a noted industrialist, makes concrete use of frequency curves in the study of several practical problems arising in the management of French firms.

A STUDY IN THE THEORY OF ECONOMIC EVOLUTION. By T. Haavelmo, North-Holland Publishing Company, Amsterdam, 1954, 114 pp.

This book is one of a series of previously unpublished studies, for the greater part by Dutch authors. The central theme of this study is a theoretical exploration of the problem of long-range economic development with particular attention to interregional differences in economic development. The author searches for possible explanations of why in fact different regions have disparate growth patterns. In this exploration the foremost variables are some index of total productive output of a region, the size of the regional population, some index of the total of accumulated capital, and some index of the level of education and technical knowledge. The book is divided into six parts: Part I, Introduction; Part II, Simple Models of Economic Growth; Part III, Deterministic Theories of Evolutionary Dissimilarities; Part IV, The Stochastic Approach; Part V, Theories of Interregional Relations; and Part VI, Some Speculations Upon a More Flexible Theoretical Framework.

THE BALANCE BETWEEN CENTRALIZATION AND DECENTRALIZATION IN MANAGERIAL CONTROL. Edited by H. J. Kruisinga, H. E. Stenfert N.V., Leiden, 1954, 118 pp.

This volume contains the proceedings of an international Study Conference organized by the Netherlands School of Economics at Rotterdam. The main purpose of the conference was to bring together well-known experts in this field and to compare the ways in which this class of problems is being tackled in various countries. The study concentrates on problems relating to the authority relationships existing among the various management levels of an organization. Problems of the geographical dispersion of the activities of an organization are not discussed. The book is divided into four parts: Part I, General Outline of the Problem, with papers by H. J. Kruisinga and E. F. L. Brech; Part II, Centralization Versus Decentralization in Private Enterprise with papers by E. Dale, H. J. van der Schroeff, and W. Vonk; Part III, Centralization

Versus Decentralization in Government Organization with papers by J. R. Simpson, R. Grégoire, and D. Simons; and Part IV, The Human, Psychological and Social Aspects Connected with the Problem of Centralization Versus Decentralization, with papers by G. L. Lippitt, J. van Vucht Tijssen, and H. W. Ouweleen.

AGGREGATE ECONOMIC ANALYSIS. By Joseph P. McKenna, Dryden, 1955, 244 pp.

This book is designed to teach people how to understand and use the analytical tools that originated with Keynes' General Theory. It assumes (1) that many who are genuinely interested in aggregate analysis find algebraic symbols confusing; (2) that the system of aggregate analysis has been unnecessarily obscured in polemical analytical articles, designed to convince economists rather than instruct students; and (3) that the system of analysis can explain much ecomic activity whether or not students accept certain policy conclusions that have been associated with it. Accordingly, Professor McKenna has translated traditional algebraic terms into numerical examples and graphic presentations and has tried to teach the reader analytical techniques rather than indoctrinate him in the "best" solutions to current economic problems. The first eleven chapters are devoted to the Keynesian system as such; the remaining five to aggregate supply, neoclassical analytical tools, and policy problems.

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